Spectral Analysis for Linear Differential Operators in Mathematical Physics

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Preface

This thesis consists of three parts.

In chapter 1, we study the spectral and pseudospectral properties of the differential operator $H_\epsilon = -\partial_x^2 + x^{2m} + i\epsilon^{-1}f(x)$ on Hilbert space $L^2(\mathbb{R})$, where $\epsilon > 0$ is a small parameter, $m \in \mathbb{N}$ and f is a real-valued Morse function which satisfies $|\partial_x^l(f(x) - |x|^{-k})| \le C|x|^{-k-l-1}$ for $l = 0, 1, 2, 3$ and large $|x|$. We show that $\Psi(\epsilon) = (\sup_{\lambda\in\mathbb{R}} \|(H_\epsilon - i\lambda)^{-1}\|)^{-1}$ and $\Sigma(\epsilon) = \inf \Re(\sigma(H_\epsilon))$ satisfy $C^{-1}\epsilon^{-\nu(m)} \le \Psi(\epsilon) \le C\epsilon^{-\nu(m)}$ and $\Sigma(\epsilon) \ge C^{-1}\epsilon^{-\nu(m)}$, $\nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}$. This extends the result of I.Gallagher, T.Gallay and F.Nier [6] (2009) for the case $m = 1$ to general $m \in \mathbb{N}$. The result of this chapter is taken form [1].

In chapter 2, we consider d-dimensional time dependent Schrödinger equations $i\partial_t u = H(t)u$, $H(t) = -(\partial_x - iA(t,x))^2 + V(t,x)$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. We assume $V(t,x)$ and $A(t,x)$ are almost critically singular with respect to spatial variables $x \in \mathbb{R}^d$ both locally and at infinity for the operator $H(t)$ to be essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. In particular, when magnetic fields $B(t,x)$ produced by $A(t,x)$ are very strong at infinity, $V(t,x)$ can explode to the negative infinity like $-\theta|B(t,x)| - C(|x|^2+1)$ for some $\theta < 1$ and $C > 0$. We show that equations uniquely generate unitary propagators in $\mathcal{H}$ under suitable conditions on the size and singularities of time derivatives of potentials $\dot{V}(t,x)$ and $\dot{A}(t,x)$. The result of this chapter is taken from [3].

In chapter 3, we consider the massless Dirac operator $H = \alpha \cdot D + Q(x)$ on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$, where $Q(x)$ is a 4×4 Hermitian matrix valued function which decays at infinity. We show that the zero resonance is absent for H, extending recent results of Y. Saitō-T. Umeda [21] and Y. Zhong -G. Gao [30]. The result of this chapter is taken from [2].

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Contents

Chapter 1

A remark on spectral properties of certain non-selfadjoint Schödinger operators.

1.1 Introduction.

We consider the following one-dimensional Schrödinger operator with a complex potential

$$\tilde{H}_\epsilon = -\partial_x^2 + x^2 + \frac{i}{\epsilon} f(x), \quad x \in \mathbb{R}$$

acting on Hilbert space $L^2(\mathbb{R})$ where $\epsilon > 0$ is a small parameter and $f(x)$ is a real-valued function. In [24, ICM], C.Villani asked the following question:

Problem 1.2. *What is the condition on $f(x)$ for $\tilde{\Sigma}(\epsilon) := \inf \Re(\sigma(\tilde{H}_\epsilon)) \to +\infty$ as $\epsilon \to 0$ and how the growth rate of divergence ?*

In [22], J.H.Schenker has proved the following theorem.

Theorem 1.2.1. *If $L_t := \{x \in \mathbb{R}; f(x) = t\}$ is essentially nowhere dense for each $t \in \mathbb{R}$, then $\tilde{\Sigma}(\epsilon) \to +\infty$ as $\epsilon \to 0$.*

Here, we say that a set S is essentially nowhere dense if $S = S' \cup N$ where S' is nowhere dense and N has Lebesgue measure zero. In [6], I.Gallagher, T.Gallay and F.Nier have studied the growth rate of $\tilde{\Sigma}(\epsilon)$ and the spectral quantity $\tilde{\Psi}(\epsilon) := \left(\sup_{\lambda\in\mathbb{R}} \|(\tilde{H}_\epsilon - i\lambda)^{-1}\|\right)^{-1}$ under the condition that $f(x)$ is a real-valued Morse function.

In this paper, we study the same problem for

$$H_\epsilon = -\partial_x^2 + x^{2m} + \frac{i}{\epsilon} f(x),\ x \in \mathbb{R},$$

where $m \geq 1$ is an integer. We shall examine how the results [6] will be changed or unchanged if the increasing rate $m \in \mathbb{N}$ of the real part of the potential is changed. We begin observing some properties of the operator H_ϵ. It is well known that the operator $H_\infty := -\partial_x^2 + x^{2m}$ is self-adjoint in $L^2(\mathbb{R})$ with domain $\mathcal{D} = \{u \in H^2(\mathbb{R}); x^{2m}u \in L^2(\mathbb{R})\}$. We consider the operator H_ϵ with same domain $\mathcal{D}$. The operator H_∞ has a compact resolvent and the spectrum $\sigma(H_\infty)$ consists of a countable number discrete positive eigenvalues. By virtue of the classical perturbation theory, the operator H_ϵ also has a compact resolvent for all $\epsilon > 0$, and the spectrum $\sigma(H_\epsilon)$ again consists of a countable number discrete eigenvalues $\{\lambda_n(\epsilon)\}_{n\in\mathbb{N}}$ with $\Re(\lambda_n(\epsilon)) \to +\infty$ as $n \to \infty$. The numerical range

$$\Theta(H_\epsilon) = \{\langle H_\epsilon u, u\rangle_{L^2}; u \in \mathcal{D}, \|u\|_{L^2} = 1\}$$

is obviously contained in the rectangle

$$\mathcal{R}_\epsilon = \{\lambda \in \mathbb{C}; \Re(\lambda) \geq a_0, \epsilon\Im(\lambda) \in \overline{f(\mathbb{R})}\}$$

where $a_0 > 0$ is the lowest eigenvalue of the self-adjoint operator H_∞. Hence, for all $n \in \mathbb{N}$ and all $\epsilon > 0$, we have

$$\lambda_n(\epsilon) \in \Theta(H_\epsilon) \subset \mathcal{R}_\epsilon.$$

It follows that the imaginary axis $i\mathbb{R}$ is contained in the resolvent set of H_ϵ. We define

$$\Sigma(\epsilon) = \inf \Re(\sigma(H_\epsilon)) = \min_{n\in\mathbb{N}} \Re(\lambda_n(\epsilon)), \quad \Psi(\epsilon) = \left(\sup_{\lambda\in\mathbb{R}} \|(H_\epsilon - i\lambda)^{-1}\|\right)^{-1}.$$

Then, we remark that

$$\Sigma(\epsilon) \geq \Psi(\epsilon) \geq a_0. \tag{1.1}$$

Indeed, for any $\lambda \in \mathbb{R}$, we have

$$\frac{1}{\operatorname{dist}(i\lambda, \sigma(H_\epsilon))} = R_{sp}((H_\epsilon - i\lambda)^{-1}) \leq \|(H_\epsilon - i\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(i\lambda, \Theta(H_\epsilon))},$$

where R_{sp} is the spectral radius. Thus, taking the supremum over $\lambda \in \mathbb{R}$, we have

$$\frac{1}{\Sigma(\epsilon)} \le \frac{1}{\Psi(\epsilon)} \le \sup_{\lambda\in\mathbb{R}} \frac{1}{\mathrm{dist}(i\lambda, \mathcal{R}_\epsilon)} = \frac{1}{a_0},$$

which implies (1.1)

Since $\Theta(H_\epsilon) \subset \mathcal{R}_\epsilon$, $H_\epsilon - a_0$ is maximal accretive and by the Hille-Yoshida theorem, it follows that $\|e^{-tH_\epsilon}\| \le e^{-a_0 t} \le 1$, for all $t \ge 0$. In [6], I. Gallagher, T. Gallay and F. Nier have shown that the decay rate of the semigroup e^{-tH_ϵ} can be controlled by the information $\Sigma(\epsilon)$ and $\Psi(\epsilon)$ as follows. The following Lemma will be proved in Appendix. Here, we corrected their mistakes.

Lemma 1.2.2. *Let A be a maximal accretive operator in a Hilbert space X. Suppose that the numerical range $\Theta(A) := \{\langle Au, u\rangle_X; u \in \mathcal{D}(A), \|u\|_X = 1\}$ is contained in the sector $\{z \in \mathbb{C}; |\mathrm{arg} z| \le \frac{\pi}{2} - 2\alpha\}$ for some $\alpha \in (0, \frac{\pi}{4}]$ and A is invetible. Define*

$$\Sigma = \inf \Re(\sigma(A)), \quad \Psi = \left(\sup_{\lambda\in\mathbb{R}} \|(A - i\lambda)^{-1}\| \right)^{-1}.$$

Then the following holds:

(i) *If there exist $C \ge 1$, $\mu > 0$ such that $\|e^{-tA}\| \le Ce^{-\mu t}$ for all $t \ge 0$, then*

$$\Sigma \ge \mu, \quad \Psi \ge \frac{\mu}{1 + \log(C)}.$$

(ii) *For any $0 < \mu < \Sigma$, we have $\|e^{-tA}\| \le C(A, \mu)e^{-\mu t}$ for all $t \ge 0$, where*

$$C(A, \mu) = \frac{1}{\pi} \left\{ \frac{\mu}{\tan\alpha} N(A, \mu) + \frac{2\pi}{\sin\alpha} \right\}, \quad N(A, \mu) = \sup_{\lambda\in\mathbb{R}} \|(A - \mu - i\lambda)^{-1}\|.$$

(iii) *If, moreover, $0 < \mu < \Psi$, then*

$$N(A, \mu) \le \frac{1}{\Psi - \mu}.$$

For the case $m = 1$, I. Gallagher, T. Gallay and F. Nier [6] have studied the lower bound for $\tilde{\Sigma}(\epsilon)$ by using a variational method "hypocoercivity" which has developed by C. Villani [23].

Theorem 1.2.3. *Suppose that* $f \in C^3(\mathbb{R})$ *satisfies* $f^{''}, f^{'''} \in L^\infty(\mathbb{R})$, *and there exist* $C_1 > 0$ *and* $0 < \nu \leq \frac{1}{2}$ *such that*

$$\langle \hat{H}_\epsilon u, u\rangle := \int_{\mathbb{R}} \left(|\partial_x u|^2 + x^{2m}|u|^2 + \frac{1}{\epsilon^2} f^{'}(x)^2 |u|^2 \right) dx \geq \frac{C_1}{\epsilon^{2\nu}} \|u\|^2 \tag{1.2}$$

for all $u \in \mathcal{D}$ *and all* $0 < \epsilon \ll 1$. *Then there exists* $C_2 > 0$ *such that, for all* $0 < \epsilon \ll 1$,

$$\Sigma(\epsilon) \geq \frac{C_2}{\epsilon^\nu}, \;\; \Psi(\epsilon) \geq \frac{C_2}{\epsilon^\nu \log(2/\epsilon)}. \tag{1.3}$$

To state our main theorem, we set the following assumption.

Assumption 1.2.4. *Assume that* $f \in C^3(\mathbb{R}, \mathbb{R})$ *has the following properties:*

(i) *All critical points of* f *are nondegenerate, i.e.,* $f^{'}(x) = 0$ *implies* $f^{''}(x) \neq 0$,

(ii) *There exist positive constants* C *and* k *such that, for all* $x \in \mathbb{R}$ *with* $|x| \geq 1$,

$$\left| \partial_x^l \left(f(x) - \frac{1}{|x|^k} \right) \right| \leq \frac{C}{|x|^{k+l+1}}, \quad for \quad l = 0, 1, 2, 3.$$

Loosely speaking, we consider the Morse functions which are bounded together with their derivatives up to the third order, and which decay like $|x|^{-k}$ at infinity.

Under the Assumption 1.2.4, it is straightforward to estimate the lowest eigenvalue of the self-adjoint operator $\hat{H}_\epsilon$. The following lemma is also a generalization of Lemma 1.7 of [6].

Lemma 1.2.5. *Suppose that* f *satisfies Assumption 1.2.4. Then there exists* $C \geq 1$ *such that, for all* $0 < \epsilon \ll 1$,

$$\frac{1}{C\epsilon^{2\rho(m)}} \leq \inf \sigma(\hat{H}_\epsilon) \leq \frac{C}{\epsilon^{2\rho(m)}}, \;\; where \;\; \rho(m) = \min\left\{ \frac{m}{k+m+1}, \frac{1}{2} \right\}.$$

The proof of this lemma will be given in Appendix. We remark that under the Assumption 1.2.4, the inequality (1.2) is satisfied with $\nu = \rho(m)$ by Lemma 1.2.5. Thus, if f satisfies the Assumption 1.2.4, we have

$$\Sigma(\epsilon) \geq \frac{C}{\epsilon^{\rho(m)}}, \quad \Psi(\epsilon) \geq \frac{C}{\epsilon^{\rho(m)} \log(2/\epsilon)}. \tag{1.4}$$

For the case $m = 1$, by modifying the proof of Theorem 1.2.3, I. Gallagher, T. Gallay and F. Nier [6] improved the exponent $\rho(m)$, if f satisfies the Assumption 1.2.4. We also extend this result for the case general $m \in \mathbb{N}$.

Theorem 1.2.6. *Suppose that f satisfies Assumption 1.2.4. Then there exists $C > 0$ such that, for all $0 < \epsilon \ll 1$,*

$$\Sigma(\epsilon) \geq \frac{C}{\epsilon^{\nu(m)}}, \quad \Psi(\epsilon) \geq \frac{C}{\epsilon^{\nu(m)} \log(2/\sqrt{\epsilon})}, \text{ where } \nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}. \tag{1.5}$$

We remark that since $\nu(m) > \rho(m)$ for all $m \in \mathbb{N}$, the lower bound in (1.5) is strictly better than in (1.4). However, Theorem 1.2.6 cannot give optimal estimates for f satisfying the Assumption 1.2.4. In fact, as we shall see below, we can remove the logarithmic term in (1.5) by using the localization techniques and semicalssical subelliptic estimates. In particular, we give an optimal estimate for $\Psi(\epsilon)$. The following theorem is main result of this paper.

Theorem 1.2.7. *Suppose that f satisfies Assumption 1.2.4. Then there exists $C \geq 1$ such that, for all $0 < \epsilon \ll 1$,*

$$\frac{1}{C\epsilon^{\nu(m)}} \leq \Psi(\epsilon) \leq \frac{C}{\epsilon^{\nu(m)}}.$$

Remark 1.2.8. *For the case $m = 1$, Theorem 1.2.7 was proven by I.Gallagher, T.Gallay and F.Nier [6]. Our result shows that $\nu(m) > \nu(n)$ if $m > n$.*

As we already remarked in (1.1), we know that $\Sigma(\epsilon) \geq \Psi(\epsilon)$. However, the following theorem shows that $\Sigma(\epsilon)$ can be much bigger than $\Psi(\epsilon)$ in some particular cases. We remark that for self-adjoint operators, $\Sigma = \Psi$ by virtue of the spectral theorem, where Σ, Ψ defined in Lemma 1.2.2. The following is also a generalization of the Theorem 1.9 of [6].

Theorem 1.2.9. *Fix $k > 0$ and set $f(x) = (1 + x^2)^{-k/2}$. Then there exists a constant $C > 0$ such that for all $0 < \epsilon \ll 1$,*

$$\Sigma(\epsilon) \geq \frac{C}{\epsilon^{\nu'(m)}}, \text{ where } \nu'(m) = \min\left\{\frac{2m}{k+2m}, \frac{1}{2}\right\}.$$

The rest of the paper is organized as follows. In Section 1.3, by using a variational method, we prove Theorem 1.2.3 and Theorem 1.2.6. In Section 1.4, by using the localization techniques and semiclassical subelliptic estimates, we prove Theorem 1.2.7. Theorem 1.2.9 is proved in Section 1.5. Before going into the next, we remark that

(i) $\Psi(\epsilon) > a_0$ if $f \in L^\infty(\mathbb{R})$ is not a constant,

(ii) $\Psi(\epsilon) \to \infty$ as $\epsilon \to 0$ if $f \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ and for any $t \in \mathbb{R}$, L_t has empty interiors.

This can be proven similarly to Proposition 1.4 and Lemma 2.1 of [6]. Throughout this paper, we denote by C various constants whose exact values are not important. Thus they may differ from one place to the other.

1.3 Variational Estimates

In this section, we prove Theorem 1.2.3 by using a variational method. For the case $m = 1$, I. Gallagher, T. Gally and F. Nier [6] have studied the lower bound of $\tilde{\Sigma}(\epsilon)$ and $\tilde{\Psi}(\epsilon)$ by applying a variational method hypocoercivity developed by C. Villani [23]. We consider the linear operator of the form $L = A^*A + B$ in a Hilbert space X where A is the linear operator and B is the skew symmetric operator. The method of hypocoercivity allows us to compare the spectral properties of L with the associated self-adjoint operator $\hat{L} = A^*A + C^*C$ where $C = [A, B] := AB - BA$. For the case $m = 1$, by setting $X = L^2(\mathbb{R})$, $A = -\partial_x + x$ and $B = (i/\epsilon)f(x)$, then we have $A^*A + B = \tilde{H}_\epsilon - 1$. On the other hand, since $C = [A, B] = (i/\epsilon)f^{'}(x)$, the associated self-adjoint operator $\hat{\tilde{H}}_\epsilon$ is defined by $A^*A + C^*C = \hat{\tilde{H}}_\epsilon - 1$ has the explicit form $\hat{\tilde{H}}_\epsilon = -\partial_x^2 + x^2 + (1/\epsilon^2)f^{'}(x)^2$. For the case general $m \in \mathbb{N}$, the method of hypocoercivity cannot be applied. However, by considering the similar operator

$$\hat{H}_\epsilon = -\partial_x^2 + x^{2m} + \frac{1}{\epsilon^2}f^{'}(x)^2,$$

we obtain the lower bound of $\Sigma(\epsilon)$ and $\Psi(\epsilon)$.

1.3.1 Proof of Theorem 1.2.3

Suppose that $f \in C^3(\mathbb{R})$ satisfies $f^{''}, f^{'''} \in L^\infty(\mathbb{R})$ and the inequality (1.2) holds for some $0 < \nu \leq 1/2$. Let $u(x,t)$ be a solution of the equation:

$$\partial_t u(x,t) = -H_\epsilon u(x,t) = \partial_x^2 u(x,t) - x^{2m}u(x,t) - \frac{i}{\epsilon}f(x)u(x,t). \tag{1.6}$$

We define the quadratic functional as follows:

$$\Phi(t) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|u|^2 + \frac{\alpha}{2}(|\partial_x u|^2 + x^{2m}|u|^2) + \beta\Re((\partial_x \bar{u})if^{'}u) + \frac{\gamma}{2}(f^{'})^2|u|^2 \right\}, \tag{1.7}$$

where α, β, γ are positive constants which precise values will be determined later. We assume that $4\beta^2 \leq \alpha\gamma$ so that

$$\frac{1}{2}\|u(t)\|^2 \leq \Phi(t) \leq \int_{\mathbb{R}} \left\{ \frac{1}{2}|u|^2 + \frac{3\alpha}{4}(|\partial_x u|^2 + x^{2m}|u|^2) + \frac{3\gamma}{4}(f^{'})^2|u|^2 \right\} dx, \tag{1.8}$$

for all $t \geq 0$. Then, we prove the following inequality:

$$\Phi^{'}(t) \leq -\eta\Phi(t), \ \textit{with} \ \eta = \mathcal{O}(\epsilon^{-\nu}). \tag{1.9}$$

To prove (1.9), we compute the time derivative of $\Phi(t)$. Using the identity (1.6) and after integration by parts, we obtain the following identities:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |u|^2 dx = -\int_{\mathbb{R}} (|\partial_x u + x^{2m}|u|^2|) dx, \tag{1.10}$$

$$\frac{\alpha}{2}\frac{d}{dt}\int_{\mathbb{R}} (|\partial_x u|^2 + x^{2m}|u|^2) dx = -\alpha\int_{\mathbb{R}} |\partial_x^2 u - x^{2m}u|^2 dx - \frac{\alpha}{\epsilon}\Re\int_{\mathbb{R}} i(\partial_x \bar{u})(f)^{'} u dx, \tag{1.11}$$

$$\beta\frac{d}{dt}\Re\int_{\mathbb{R}} i(\partial_x\bar{u})(f^{'})udx = -\frac{\beta}{\epsilon}\int_{\mathbb{R}} (f^{'})^2|u|^2 dx - \beta\Re\int_{\mathbb{R}} i\bar{u}f^{'''}\partial_x u dx$$
$$+2\beta\Re\int_{\mathbb{R}} i(\partial_x\bar{u})(f^{'})(\partial_x^2 u - x^{2m}u)dx, \tag{1.12}$$

$$\frac{\gamma}{2}\frac{d}{dt}\int_{\mathbb{R}} (f^{'})^2|u|^2 dx = -\gamma\int_{\mathbb{R}} (f^{'})^2(|\partial_x u|^2 + x^{2m}|u|^2)dx - 2\gamma\Re\int_{\mathbb{R}} f^{'}f^{''}(\partial_x u)\bar{u}dx. \tag{1.13}$$

To estimate the various terms in $\Phi^{'}(t)$, we define

$$L_j = \|\partial_x^j f\|_\infty, \quad j = 2, 3$$

and we use the following bounds:

$$-\frac{\alpha}{\epsilon}\Re\int_{\mathbb{R}} i(\partial_x\bar{u})f^{'}udx \leq \frac{1}{4}\int_{\mathbb{R}} |\partial_x u|^2 dx + \frac{\alpha^2}{\epsilon^2}\int_{\mathbb{R}} (f^{'})^2|u|^2 dx, \tag{1.14}$$

$$-\beta\Re\int_{\mathbb{R}} i\bar{u}f^{'''}(\partial_x u)dx \leq L_3\beta\int_{\mathbb{R}} |u(\partial_x u)|dx \leq \frac{L_3\beta(a_0+1)}{2a_0}\int_{\mathbb{R}} (|\partial_x u|^2 + x^{2m}|u|^2)dx, \tag{1.15}$$

$$2\beta\Re\int_{\mathbb{R}} i(\partial_x\bar{u})f^{'}(\partial_x^2 u - x^{2m}u)dx \leq \frac{\alpha}{2}\int_{\mathbb{R}} |\partial_x^2 u - x^{2m}u|^2 dx + \frac{2\beta^2}{\alpha}\int_{\mathbb{R}} (f^{'})^2|\partial_x u|^2 dx, \tag{1.16}$$

$$-2\gamma\Re\int_{\mathbb{R}} f^{'}f^{''}\bar{u}(\partial_x u)dx \leq \frac{1}{4}\int_{\mathbb{R}} |\partial_x u|^2 dx + 4\gamma^2 L_2^2\int_{\mathbb{R}} (f^{'})^2|u|^2 dx. \tag{1.17}$$

Using the estimate (1.10)-(1.17), we have

$$\Phi'(t) \le \left(-\frac{1}{2} + \frac{L_3\beta(a_0+1)}{2a_0}\right)\int_{\mathbb{R}}(|\partial_x u|^2 + x^{2m}|u|^2)dx$$
$$-\frac{\alpha}{2}\int_{\mathbb{R}}|\partial_x^2 u - x^{2m}u|^2dx + \left(-\frac{\beta}{\epsilon} + \frac{\alpha^2}{\epsilon^2} + 4\gamma^2L_2^2\right)\int_{\mathbb{R}}(f')^2|u|^2dx$$
$$+\left(-\gamma + \frac{2\beta^2}{\alpha}\right)\int_{\mathbb{R}}(f')^2(|\partial_x u|^2 + x^{2m}|u|^2)dx. \tag{1.18}$$

We now define positive constants α, β and γ as follows:

$$\alpha = \left(\frac{\beta\epsilon}{4}\right)^{\frac{1}{2}},\ \beta = \min\left\{\frac{a_0}{2L_3(a_0+1)}, \frac{1}{32L_2}\right\},\ \gamma = 8\left(\frac{\beta^3}{\epsilon}\right)^{\frac{1}{2}}. \tag{1.19}$$

Then, it is obvious that $4\beta^2 = \alpha\gamma$, $4\gamma^2L_2^2 \le \beta/4\epsilon$, and we thus have

$$\Phi'(t) \le -\frac{1}{4}\int_{\mathbb{R}}(|\partial_x u|^2 + x^{2m}|u|^2)dx - \frac{\beta}{2\epsilon}\int_{\mathbb{R}}(f')^2|u|^2dx \tag{1.20}$$
$$-\frac{\alpha}{2}\int_{\mathbb{R}}|\partial_x^2 u - x^{2m}u|^2dx - \frac{\gamma}{2}\int_{\mathbb{R}}(f')^2(|\partial_x u|^2 + x^{2m}|u|^2)dx. \tag{1.21}$$

We neglect both terms in (1.21) and use only the upper bound (1.20). We may assume $\epsilon \le 1/(2\beta)$ without losing generality. Using (1.20) and (1.2), we obtain that

$$\Phi'(t) \le -\frac{1}{8}\int_{\mathbb{R}}(|\partial_x u|^2 + x^{2m}|u|^2)dx - \frac{\beta}{4\epsilon}\int_{\mathbb{R}}(f')^2|u|^2dx - \frac{\beta C_1}{4}\frac{(2\beta)^{\nu+1}}{\epsilon^\nu}\int_{\mathbb{R}}|u|^2dx. \tag{1.22}$$

Thus, combining (1.8) and (1.22), we have

$$\Phi'(t) \le -\eta\Phi(t)\ \textit{with}\ \eta = \min\left\{\frac{1}{6\alpha}, \frac{\beta}{3\epsilon\gamma}, \frac{\beta C_1(2\beta)^{\nu+1}}{2\epsilon^\nu}\right\}, \tag{1.23}$$

which proves (1.9).

We next deduce some information of the semigroup e^{-tH_ϵ} in $L^2(\mathbb{R})$. Let $u(x,t)$ be a solution of the equation (1.6) with initial data $u_0 \in L^2(\mathbb{R})$. By integrate with respect to t in (1.10), we have

$$\int_0^{\sqrt{\epsilon}} \left\{ \|\partial_x u(t)\|^2 + \|x^m u(t)\|^2 \right\} dt = \frac{1}{2} \left(\|u_0\|^2 - \|u(\sqrt{\epsilon})\|^2 \right) \le \frac{1}{2} \|u_0\|^2. \tag{1.24}$$

Then, there exists $0 < \tau \le \sqrt{\epsilon}$ such that

$$\|u(\tau)\|^2 \le \|u_0\|^2, \ \|\partial_x u(\tau)\|^2 + \|x^m u(\tau)\|^2 \le \frac{\|u_0\|^2}{2\sqrt{\epsilon}}, \tag{1.25}$$

here, we used the fact that $\|u(t)\|^2$ is monotone decreasing function of t (see (1.10)). Using (1.7), (1.19), (1.25) and the fact that $|f^{'}(x)| \le |f^{'}(0)| + L_2(1 + |x|^m)$ for all $x \in \mathbb{R}$, there exists $C > 0$ such that for any $0 < \epsilon \ll 1$,

$$\Phi(\sqrt{\epsilon}) \le \Phi(\tau) \le \frac{C}{\epsilon} \|u_0\|^2. \tag{1.26}$$

Hence, we have

$$\|u(t + \sqrt{\epsilon})\|^2 \le 2\Phi(t + \sqrt{\epsilon}) \le 2e^{-\eta t} \Phi(\sqrt{\epsilon}) \le \frac{2C}{\epsilon} e^{-\eta t} \|u_0\|^2, \ for \ all \ t \ge 0,$$

and it follows that

$$\|e^{-(t+\sqrt{\epsilon})H_\epsilon}\| \le \left(\frac{2C}{\epsilon} \right)^{\frac{1}{2}} e^{-\frac{C_2 t}{\epsilon^\nu}}, \ for \ all \ t \ge 0.$$

By virtue of Lemma 1.2.2 (i), we have

$$\Sigma(\epsilon) \ge \frac{C_2}{\epsilon^\nu}.$$

Since $\nu \le 1/2$ and $\|e^{-tH_\epsilon}\| \le 1$ for all $t \ge 0$, we have

$$\begin{aligned} \|(H_\epsilon - i\lambda)^{-1}\| &\le \int_0^\infty \|e^{-tH_\epsilon}\| dt \le \int_0^{\sqrt{\epsilon}} dt + \int_0^\infty \min \left\{ 1, \left(\frac{2C}{\epsilon} \right)^{\frac{1}{2}} e^{-\frac{C_2 t}{\epsilon^\nu}} \right\} dt \\ &\le \epsilon^\nu + \frac{\epsilon^\nu}{C_2} \left\{ \log \left(\frac{2C}{\epsilon} \right)^{\frac{1}{2}} + 1 \right\}, \ for \ all \ \lambda \in \mathbb{R}. \end{aligned}$$

Thus, taking the supremum over $\lambda \in \mathbb{R}$, we have

$$\Psi(\epsilon) \ge \frac{C}{\epsilon^\nu \log \left(\frac{2}{\epsilon} \right)}.$$

This completes the proof of Theorem 1.2.3.

1.3.2 Improved decay rate under the Assumption 1.2.4-Proof of Theorem 1.2.6

For the case $m = 1$, under the Assumption 1.2.4, by considering the functional $\Phi(t)$ defined in (1.7) with the parameters α, β, γ depending on the space variable x, I. Gallagher, T. Gallay and F. Nier [6] have improved the exponent ν in Theorem 1.2.3. In this section, we generalize their result to the case general $m \in \mathbb{N}$.

Let $u(x,t)$ be a solution of the equation (1.6). We again consider the quadratic functional $\Phi(t)$ defined by (1.7). However, now α, β, γ are positive functions of space variable x. Let $\beta_0 > 0$ be a small constant whose precise value will be determined later. Take a large enough constant $A > 0$ so that all critical points of f are contained in the interval $[-A+1, A-1]$. Then, we define functions $\alpha(x), \beta(x), \gamma(x)$ as follows:

$$\alpha(x) = \left(\frac{\beta(x)\epsilon}{4}\right)^{1/2}, \gamma(x) = 8\left(\frac{\beta(x)^3}{\epsilon}\right)^{1/2} \tag{1.27}$$

$$\beta(x) = \begin{cases} \beta_0, & \text{if} \quad |x| \le A, \\ \beta_0 \left(\frac{|x|}{A}\right)^{k-m+1}, & \text{if} \quad A \le |x| \le B_\epsilon, \\ \beta_0 \epsilon^{-\frac{k-m+1}{k+3m+1}}, & \text{if} \quad |x| \ge B_\epsilon, \qquad B_\epsilon := A\epsilon^{-\frac{1}{k+3m+1}} \end{cases} \tag{1.28}$$

where k is as in the Assumption 1.2.4. It is easy to verify that the function $\beta(x)$ has the following properties: for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, all $x \in \mathbb{R}$

$$\epsilon^{\frac{1}{2}}\beta'(x)^2 \le \delta\beta(x)^{\frac{3}{2}}, \ \epsilon\beta(x) \le \delta, \ \epsilon\beta'(x)^2 \le \delta\beta(x). \tag{1.29}$$

We again prove the following inequality as in the proof of Theorem 1.2.3.

$$\Phi'(t) \le -\eta\Phi(t), \ \textit{with} \ \eta = \mathcal{O}(\epsilon^{-\nu(m)}), \ \nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}. \tag{1.30}$$

We compute the time derivative of $\Phi(t)$. We shall only point out what modifications are necessary in Theorem 1.2.3. The equation (1.10) is unchanged. The equation (1.11) is changed as follows:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \alpha(x)(|\partial_x u|^2 + x^{2m}|u|^2)dx = -\int_{\mathbb{R}} \alpha(x)|\partial_x^2 u - x^{2m} u|^2 dx$$

$$-\frac{1}{\epsilon}\Re\int_{\mathbb{R}} i\alpha(x)(\partial_x\bar{u})(f)^{'}udx - \Re\int_{\mathbb{R}}\alpha^{'}(x)(\partial_x\bar{u})(\partial_x^2 u - x^{2m}u)dx. \qquad (1.31)$$

Using (1.29), we have for small $\epsilon > 0$,

$$\frac{\alpha^{'}(x)^2}{\alpha(x)} = \frac{\epsilon^{1/2}\beta^{'}(x)^2}{8\beta(x)^{3/2}} \leq \frac{1}{12}, \quad for\ all\ x \in \mathbb{R}.$$

The last term of right hand side (1.31) is bounded as follows:

$$\begin{aligned}-\Re\int_{\mathbb{R}}\alpha^{'}(x)(\partial_x\bar{u})(\partial_x^2 u - x^{2m}u)dx &\leq \frac{1}{4}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + \int_{\mathbb{R}}\frac{\alpha^{'}(x)^2}{\alpha(x)}|\partial_x u|^2dx\\ &\leq \frac{1}{4}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + \frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx.\end{aligned}$$

Since $\alpha(x)^2 = \beta(x)\epsilon/4$, the second term of right hand side (1.31) is bounded as follows:

$$-\frac{1}{\epsilon}\Re\int_{\mathbb{R}} i\alpha(x)(\partial_x\bar{u})f^{'}udx \leq \frac{1}{4}\int_{\mathbb{R}}|\partial_x u|^2dx + \frac{1}{4\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx.$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\alpha(x)(|\partial_x u|^2 + x^{2m}|u|^2)dx \leq &-\frac{3}{4}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx\\ &+\frac{1}{3}\int_{\mathbb{R}}|\partial_x u|^2dx + \frac{1}{4\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx.\end{aligned} \qquad (1.32)$$

The equation (1.12) is changed as follows:

$$\begin{aligned}&\frac{d}{dt}\Re\int_{\mathbb{R}} i\beta(x)(\partial_x\bar{u})(f^{'})udx = -\frac{1}{\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx - \Re\int_{\mathbb{R}} i\beta(x)\bar{u}f^{'''}\partial_x udx\\ &+2\Re\int_{\mathbb{R}} i\beta(x)(\partial_x\bar{u})(f^{'})(\partial_x^2 u - x^{2m}u)dx + \Re\int_{\mathbb{R}} i\beta^{'}(x)\bar{u}f^{'}(\partial_x^2 u - x^{2m}u)dx\\ &-\Re\int_{\mathbb{R}} i\beta^{'}(x)\bar{u}f^{''}(\partial_x u)dx.\end{aligned} \qquad (1.33)$$

Since $\beta^{'}(x)^2/\alpha(x) \ll \beta(x)/\epsilon$ for small enough $\epsilon > 0$, the forth term of right hand side (1.33) is bounded as follows:

$$\Re\int_{\mathbb{R}} i\beta'(x)\bar{u}f'(\partial_x^2 u - x^{2m}u)dx \le \frac{1}{8}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + 2\int_{\mathbb{R}}\frac{\beta'(x)^2}{\alpha(x)}(f')^2|u|^2dx$$
$$\le \frac{1}{8}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx, \quad (1.34)$$

Similarly, the fifth term of right hand side (1.33) is bounded as follows:

$$\Re\int_{\mathbb{R}} i\beta'(x)\bar{u}f''(\partial_x u)dx \le \frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx + 3\int_{\mathbb{R}}\beta'(x)^2(f'')^2|u|^2dx$$
$$\le \frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx + \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx, \quad (1.35)$$

where in the last inequality we used (1.29) and the fact that $|f''(x)| \le C_k|f'(x)|$ for all $x \in \operatorname{supp}\beta'$. Since $\alpha(x)^2 = \beta(x)\epsilon/4$, the third term of right hand side (1.33) is bounded as follows:

$$2\Re\int_{\mathbb{R}} i\beta(x)(\partial_x\bar{u})f'(\partial_x^2 u - x^{2m}u)dx \le \frac{1}{2}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + \frac{1}{2}\int_{\mathbb{R}}\gamma(x)(f')^2|\partial_x u|^2dx.$$

It remains to estimate the second term of right hand side (1.33). We first note that

$$-\Re\int_{\mathbb{R}} i\beta(x)\bar{u}f'''(\partial_x u)dx \le \frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx + 3\int_{|x|\le A}\beta(x)^2(f''')^2|u|^2dx$$
$$+ 3\int_{|x|\ge A}\beta(x)^2(f''')^2|u|^2dx.$$

We choose $\beta_0 > 0$ so that $3\beta_0^2L_3^2/a_0 \le 1/6$. Then, we have

$$3\int_{|x|\le A}\beta(x)^2(f''')^2|u|^2dx \le 3\beta_0^2L_3^2\int_{\mathbb{R}}|u|^2dx \le \frac{1}{6}\int_{\mathbb{R}}(|\partial_x u|^2 + x^{2m}|u|^2)dx.$$

Using (1.29) and the fact that $|f'''(x)| \le C_k|f'(x)|$ for all $|x| \ge A$, we have for small enough $\epsilon > 0$,

$$3\int_{|x|\ge A}\beta(x)^2(f''')^2|u|^2dx \le \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx.$$

Summarizing, we thus have

$$\frac{d}{dt}\Re\int_{\mathbb{R}} i\beta(x)f^{'}u(\partial_x\bar{u})dx \leq -\frac{3}{4\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx + \frac{1}{3}\int_{\mathbb{R}}(|\partial_x u|^2 + x^{2m}|u|^2)dx + \frac{5}{8}\int_{\mathbb{R}}\alpha(x)|\partial_x^2 u - x^{2m}u|^2dx + \frac{1}{2}\int_{\mathbb{R}}\gamma(x)(f^{'})^2|\partial_x u|^2dx. \quad (1.36)$$

Finally, we consider (1.13). The equation (1.13) is changed as follows:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\gamma(x)(f^{'})^2|u|^2dx = -\int_{\mathbb{R}}\gamma(x)(f^{'})^2(|\partial_x u|^2 + x^{2m}|u|^2)dx - \Re\int_{\mathbb{R}}\gamma^{'}(x)(f^{'})^2\bar{u}(\partial_x u)dx - 2\Re\int_{\mathbb{R}}\gamma(x)f^{'}f^{''}(\partial_x u)\bar{u}dx. \quad (1.37)$$

Using (1.29), we have for small enough $\epsilon > 0$, $\gamma^{'}(x)^2/\gamma(x) \ll \beta(x)/\epsilon$. Then, the second term of right hand side (1.37) is bounded as follows:

$$\begin{aligned}-\Re\int_{\mathbb{R}}\gamma^{'}(x)(f^{'})^2\bar{u}(\partial_x u)dx &\leq \frac{1}{4}\int_{\mathbb{R}}\gamma(x)(f^{'})^2|\partial_x u|^2dx + \int_{\mathbb{R}}\frac{\gamma^{'}(x)^2}{\gamma(x)}(f^{'})^2|u|^2dx\\ &\leq \frac{1}{4}\int_{\mathbb{R}}\gamma(x)(f^{'})^2|\partial_x u|^2dx + \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx.\end{aligned}$$

To estimate the third term of right hand side (1.37), we take a partition of unity $\phi_1(x)+\phi_2(x) \equiv 1$ such that $\phi_1, \phi_2 \in C^\infty(\mathbb{R})$, $\operatorname{supp}\phi_1 \subset \{|x| \leq A\}$, and $\operatorname{supp}\phi_2 \subset \{|x| \geq \frac{3}{2}A\}$. We define $\gamma_1 = \gamma\phi_1$ and $\gamma_2 = \gamma\phi_2$ and choose $\beta_0 > 0$ so that $768L_2^2\beta_0^2 2^{2|k-m+1|} \leq 1/12$. Then, we have

$$\begin{aligned}-2\Re\int_{\mathbb{R}}\gamma_1(x)f^{'}f^{''}\bar{u}(\partial_x u)dx &\leq \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx + 12\epsilon L_2^2\int_{|x|\leq 2A}\frac{\gamma(x)^2}{\beta(x)}|\partial_x u|^2dx\\ &\leq \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f^{'})^2|u|^2dx + \frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx,\end{aligned}$$

here, in the second inequality, we used $\epsilon\gamma(x)^2/\beta(x) = 64\beta(x)^2$ and the fact that $\beta(x) \leq \beta_0 2^{|k-m+1|}$ for $|x| \leq 2A$. On the other hand, integrating by parts in the third term of right hand side (1.37), we have

$$-2\Re\int_{\mathbb{R}}\gamma_2(x)f^{'}f^{''}\bar{u}(\partial_x u)dx = \int_{\mathbb{R}}\gamma_2(x)\left((f^{''})^2 + f^{'}f^{'''}\right)|u|^2dx + \int_{\mathbb{R}}\gamma_2^{'}(x)f^{'}f^{''}|u|^2dx.$$

Since $|\gamma'(x)/\gamma(x)| = \frac{3}{2}|\beta'(x)/\beta(x)| \le |k-m+1|/A$ by the definition of $\beta(x)$ and $\gamma(x)$, it is obvious that $|\gamma_2'(x)| \le C\gamma(x)$ for some $C>0$ independent of $\epsilon>0$. Using the fact that $|f'''(x)|+|f''(x)| \le C_k|f'(x)|$ for all $x \in \operatorname{supp}\gamma_2$ and $\gamma(x) \ll \beta(x)/\epsilon$ for small enough $\epsilon>0$, we have

$$-2\Re\int_{\mathbb{R}}\gamma_2(x)f'f''\bar{u}(\partial_x u)dx \le C\int_{\mathbb{R}}\gamma(x)(f')^2|u|^2dx \le \frac{1}{12\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx.$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\gamma(x)(f')^2|u|^2dx \le &-\frac{3}{4}\int_{\mathbb{R}}\gamma(x)(f')^2(|\partial_x u|^2+x^{2m}|u|^2)dx\\ &+\frac{1}{12}\int_{\mathbb{R}}|\partial_x u|^2dx+\frac{1}{4\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx. \quad (1.38)\end{aligned}$$

Summarizing the estimates (1.10), (1.32), (1.36) and (1.38), we obtain that

$$\Phi'(t) \le -\frac{1}{4}\int_{\mathbb{R}}(|\partial_x u|^2+x^{2m}|u|^2)dx-\frac{1}{4\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx \quad (1.39)$$

$$-\frac{1}{8}\int_{\mathbb{R}}\alpha(x)|\partial_x^2u-x^{2m}u|^2dx-\frac{1}{4}\int_{\mathbb{R}}\gamma(x)(f')^2(|\partial_x u|^2+x^{2m}|u|^2)dx. \quad (1.40)$$

We neglect both terms in (1.40) and use only the upper bound in (1.39) as in the proof of Theorem 1.2.3. We use the following fact which will be proved in Lemma 1.6.1 (Appendix). There exists $C>0$ such that for any $u \in \mathcal{D}$ and $0<\epsilon\ll 1$,

$$\int_{\mathbb{R}}\left(|\partial_x u|^2+x^{2m}|u|^2+\frac{\beta(x)f'(x)^2}{\epsilon}|u|^2\right)dx \ge \frac{C}{\epsilon^{\nu(m)}}\|u\|^2,\ \nu(m)=\min\left\{\frac{2m}{k+3m+1},\frac{1}{2}\right\}. \quad (1.41)$$

Hence, using (1.39) and (1.41), we obtain that

$$\Phi'(t) \le -\frac{1}{8}\int_{\mathbb{R}}(|\partial_x u|^2+x^{2m}|u|^2)dx-\frac{1}{8\epsilon}\int_{\mathbb{R}}\beta(x)(f')^2|u|^2dx-\frac{C}{8\epsilon^{\nu(m)}}\int_{\mathbb{R}}|u|^2dx,$$

and combining the upper bound estimate in (1.8), we have

$$\Phi'(t) \le -\eta\Phi(t),\ \textit{with}\ \eta=\min\left\{\frac{1}{6\|\alpha\|_{L^\infty}},\frac{1}{6\epsilon}\left\|\frac{\gamma}{\beta}\right\|_{L^\infty}^{-1},\frac{C}{4\epsilon^{\nu(m)}}\right\}=\mathcal{O}(\epsilon^{-\nu(m)}),$$

which proves (1.30). The rest of the proof can be done as in the proof of Theorem 1.2.3. Thus, we have

$$\Sigma(\epsilon) \geq \frac{C}{\epsilon^{\nu(m)}}, \quad \Psi(\epsilon) \geq \frac{C}{\epsilon^{\nu(m)} \log(2/\sqrt{\epsilon})}.$$

This completes the proof of Theorem 1.2.6.

1.4 Resolvent Estimates.

In this section, we prove Theorem 1.2.7 by using the localization techniques and semiclassical subelliptic estimates. In particular, we remove the logarithmic term in (1.5) and give an optimal estimate for $\Psi(\epsilon)$. The proof patterns after that of Proposition 4.1 of [6]. We estimate

$$\kappa(\epsilon,\lambda) := \|(H_\epsilon - i\lambda)^{-1}\|, \quad \lambda \in \mathbb{R},\ 0 < \epsilon \ll 1.$$

Under the Assumption 1.2.4, f has only a finite number of critical points. We denote the set of critical values of f by

$$\mathrm{cv}(f) = \{f(x); x \in \mathbb{R}, f^{'}(x) = 0\}.$$

Proposition 1.4.1. *If f satisfies Assumption 1.2.4, then for any $\lambda \in \mathbb{R}$ and $0 < \epsilon \ll 1$, the quantity $\kappa(\epsilon,\lambda)$ satisfies the following estimates:*

(i) *If* $\mathrm{dist}(\epsilon\lambda, f(\mathbb{R})) \geq \delta > 0$, *then* $\kappa(\epsilon,\lambda) \leq \epsilon/\delta$.

(ii) *If* $\mathrm{dist}(\epsilon\lambda, \mathrm{cv}(f) \cup \{0\}) \geq \delta > 0$, *then* $\kappa(\epsilon,\lambda) \leq C_\delta \epsilon^{2/3}$.

(iii) *If $\lambda = \lambda(\epsilon)$ is such that* $\lim_{\epsilon\to 0} \epsilon\lambda(\epsilon) = \alpha \in \mathrm{cv}(f)\backslash\{0\}$, *then*

$$\varlimsup_{\epsilon\to 0} \epsilon^{-1/2}\kappa(\epsilon,\lambda(\epsilon)) \leq C.$$

(iv) *For $\lambda = 0$, the quantity $\kappa(\epsilon, 0)$ satisfies*

$$\kappa(\epsilon,0) \leq \begin{cases} C\epsilon^{\frac{2m}{k+2m}}, & \text{if} \quad 0 \notin f(\mathbb{R}), \\ C\epsilon^{\min\left\{\frac{2m}{k+2m},\frac{2}{3}\right\}}, & \text{if} \quad 0 \in f(\mathbb{R})\backslash \mathrm{cv}(f), \\ C\epsilon^{\min\left\{\frac{2m}{k+2m},\frac{1}{2}\right\}}, & \text{if} \quad 0 \in \mathrm{cv}(f). \end{cases}$$

(v) *There exists $C > 1$ such that, for all $\lambda \in \mathbb{R}$ and $0 < \epsilon \ll 1$,*

$$\kappa(\epsilon,\lambda) \leq C\epsilon^{\nu(m)}. \quad \text{where} \quad \nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}.$$

For the proof of Proposition 1.4.1, we use the following localization scheme.

1.4.1 The localization formula

Lemma 1.4.2. *Let $Q = -\Delta + V$ in $\mathbb{R}^d$, where V is a complex valued measurable function. Let $\{\chi_j^2\}_{j\in J}$, where $\chi_j \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ be such that*

$$\sum_{j\in J} \chi_j(x)^2 = 1, \; \text{for all } x \in \mathbb{R}^d, \text{ and}$$

$$m_1^2 := \sup_{x\in\mathbb{R}^d} \sum_{j\in J} |\nabla\chi_j(x)|^2 < +\infty, \quad m_2^2 := \sup_{x\in\mathbb{R}^d} \sum_{j\in J} (\Delta\chi_j(x))^2 < +\infty. \tag{1.42}$$

Then, the following estimates hold for any $u \in C_0^\infty(\mathbb{R}^d)$

$$2\|Qu\|^2 + 3m_2^2\|u\|^2 + 8m_1^2\|\nabla u\|^2 \geq \sum_{j\in J} \|Q\chi_j u\|^2, \tag{1.43}$$

in particular, if $\Re V(x) \geq 0$,

$$2\|Qu\|^2 + 3m_2^2\|u\|^2 + 8m_1^2\Re\langle Qu, u\rangle_{L^2} \geq \sum_{j\in J} \|Q\chi_j u\|^2, \tag{1.44}$$

$$\langle Qu, u\rangle_{L^2} + m_1^2\|u\|^2 \geq \sum_{j\in J} \langle Q\chi_j u, \chi_j u\rangle_{L^2}. \tag{1.45}$$

Proof. For any $\chi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$, we have

$$\begin{aligned} Q^*\chi^2 Q &= Q^*\chi Q\chi + Q^*\chi[\chi, Q] \\ &= \chi Q^* Q\chi + [Q^*, \chi]Q\chi + Q^*\chi[\chi, Q] \\ &= \chi Q^* Q\chi + [Q^*, \chi]\chi Q + Q^*\chi[\chi, Q] + [Q^*, \chi][Q, \chi] \\ &= \chi Q^* Q\chi - [\Delta, \chi]\chi Q + Q^*\chi[\Delta, \chi] + [\Delta, \chi][\Delta, \chi]. \end{aligned}$$

Since

$$[\Delta, \chi] = 2(\nabla\chi)\cdot\nabla + (\Delta\chi) = 2\nabla\cdot(\nabla\chi) - (\Delta\chi),$$

we have

$$Q^*\chi^2 Q = \chi Q^* Q\chi - \nabla\cdot(\nabla\chi^2)Q + (\Delta\chi)\chi Q + Q^*(\nabla\chi^2)\cdot\nabla + Q^*\chi(\Delta\chi) - R_\chi^* R_\chi,$$

where $R_\chi = 2(\nabla\chi)\cdot\nabla + (\Delta\chi)$. Applying this identity to $\chi = \chi_j$ and summing over $j \in J$, we have

$$Q^*Q = \sum_{j\in J} \chi_j Q^* Q \chi_j + \sum_{j\in J} \{(\Delta\chi_j)\chi_j Q + Q^*\chi_j(\Delta\chi_j)\} - \sum_{j\in J} R^*_{\chi_j} R_{\chi_j}.$$

Since

$$\begin{aligned}
&\langle (\Delta\chi_j)\chi_j Qu + Q^*\chi_j(\Delta\chi_j)u, u\rangle \geq -\|(\Delta\chi_j)u\|^2 - \|\chi_j Qu\|^2, \\
&\langle -R^*_{\chi_j} R_{\chi_j} u, u\rangle \geq -8\|(\nabla\chi_j)\cdot\nabla u\|^2 - 2\|(\Delta\chi_j)u\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
\langle Q^*Qu, u\rangle &\geq \sum_{j\in J} \left\{ \|Q\chi_j u\|^2 - \left(\|\chi_j Qu\|^2 + \|(\Delta\chi_j)u\|^2\right) - \left(8\|(\nabla\chi_j)\cdot\nabla u\|^2 + 2\|(\Delta\chi_j)u\|^2\right)\right\} \\
&\geq \sum_{j\in J} \|Q\chi_j u\|^2 - \|Qu\|^2 - 8m_1^2\|\nabla u\|^2 - 3m_2^2\|u\|^2.
\end{aligned}$$

This implies (1.43). In particular, if $\Re V(x) \geq 0$, it is obvious that $\Re\langle Qu, u\rangle \geq \|\nabla u\|^2$. Thus (1.44) follows by (1.43). Finally, we prove the inequality (1.45). Since $[\chi_j, Q] = [\Delta, \chi_j] = 2(\nabla\chi_j)\cdot\nabla + (\Delta\chi_j)$, we have $[\chi_j, [\chi_j, Q]] = (\nabla\chi_j^2)\cdot\nabla - 2|\nabla\chi_j|^2$. Hence,

$$\sum_{j\in J} [\chi_j, [\chi_j, Q]] = -2\sum_{j\in J} |\nabla\chi_j|^2. \tag{1.46}$$

On the other hand, since $[\chi_j, [\chi_j, Q]] = \chi_j^2 Q + Q\chi_j^2 - 2\chi_j Q\chi_j$, we have

$$\sum_{j\in J} [\chi_j, [\chi_j, Q]] = 2Q - 2\sum_{j\in J} \chi_j Q\chi_j. \tag{1.47}$$

Thus, it follows from (1.46) and (1.47) that

$$\langle Qu, u\rangle + m_1^2\|u\|^2 \geq \langle Qu, u\rangle + \left\langle \sum_{j\in J} |\nabla\chi_j|^2 u, u \right\rangle = \sum_{j\in J} \langle Q\chi_j u, \chi_j u\rangle.$$

This completes the proof. □

Using a dyadic partition of unity, we apply Lemma 1.4.2 to the one-dimensional operator $Q = H_\epsilon - i\lambda$.

Lemma 1.4.3. *For $j \in \mathbb{N}$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$, we define unitary operators U_j, $j \in \mathbb{N}$ by $(U_j u)(x) = 2^{j/2} u(2^j x)$ and transform Q by U_j*

$$P_{j,\epsilon,\lambda} = -2^{-2j}\partial_x^2 + 2^{2mj}x^{2m} + \frac{i}{\epsilon} f(2^j x) - i\lambda,$$

and let

$$C_j(\epsilon,\lambda) = \inf\{\|P_{j,\epsilon,\lambda}u\|; u \in C_0^\infty(\mathbb{R}), \text{supp } u \subset K_j, \|u\| = 1\},$$

where $K_0 = [-1,1]$ and $K_j = [-1,-3/8] \cup [1,3/8]$ for any $j > 0$. Then $\kappa(\epsilon,\lambda) = \|(H_\epsilon - i\lambda)^{-1}\|$ satisfies

$$\left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^{-1} \leq \kappa(\epsilon,\lambda) \leq C\left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^{-1} \tag{1.48}$$

for some constant $C \geq 1$ independent of $\epsilon > 0, \lambda \in \mathbb{R}$.

Remark 1.4.4. *It is clear that $C_j(\epsilon,\lambda) \geq a_0$ for all $j \in \mathbb{N}$, $\epsilon > 0$, $\lambda \in \mathbb{R}$, because*

$$a_0\|u\|^2 \leq \Re\langle P_{j,\epsilon,\lambda}u, u\rangle \leq \|P_{j,\epsilon,\lambda}u\|\|u\|, \ for\ all\ u \in C_0^\infty(\mathbb{R}).$$

Proof. We first prove the upper bound in (1.48). Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a dyadic partition of unity such that

$$\sum_{j=0}^{\infty} \chi_j(x)^2 = \chi_0(x)^2 + \sum_{j=1}^{\infty} \tilde{\chi}\left(\frac{x}{2^j}\right) = 1$$

where $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R})$ satisfy

$$\chi_0(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{3}{4}\ , \\ 0, & \text{if } |x| \geq 1\ , \end{cases} \quad \widetilde{\chi}(x) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq |x| \leq \frac{3}{4}\ , \\ 0, & \text{if } |x| \leq \frac{3}{8} \text{ or } |x| \geq 1\ . \end{cases}$$

Then, it is obvious that

$$m_1^2 = \sup_{x\in\mathbb{R}^d} \sum_{j\in J} |\nabla\chi_j(x)|^2 < +\infty,\ m_2^2 = \sup_{x\in\mathbb{R}^d} \sum_{j\in J} (\Delta\chi_j(x))^2 < +\infty.$$

Thus, we apply Lemma 1.4.2 to the one-dimensional operator $Q = H_\epsilon - i\lambda$. Since $a_0\|u\|^2 \leq \Re(Qu,u) \leq \|Qu\|\|u\| \leq \frac{\|Qu\|}{a_0}$ for all $u \in C_0^\infty(\mathbb{R})$, it follows from the localization formula (1.44) that

$$C^2\|Qu\|^2 \geq \sum_{j=0}^{\infty} \|Q\chi_j u\|^2, \quad where \;\; C^2 = 2 + \frac{8m_1^2}{a_0} + \frac{3m_2^2}{a_0^2}.$$

For any $j \in \mathbb{N}$, we define

$$v_j(x) = 2^{j/2}\chi_j(2^j x)u(2^j x), \quad x \in \mathbb{R},$$

so that $\operatorname{supp} v_j \subset \operatorname{supp} \chi_j(2^j \cdot) \subset K_j$ and $(P_{j,\epsilon,\lambda}v_j)(x) = 2^{j/2}(Q\chi_j u)(2^j x)$. Then, we have

$$\begin{aligned} C^2\|Qu\|^2 &\geq \sum_{j=0}^{\infty} \|P_{j,\epsilon,\lambda}v_j\|^2 \geq \left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^2 \sum_{j=0}^{\infty} \|v_j\|^2 \\ &= \left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^2 \sum_{j=0}^{\infty} \|\chi_j u\|^2 = \left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^2 \|u\|^2. \end{aligned} \tag{1.49}$$

Since $Q = H_\epsilon - i\lambda$ and $C_0^\infty(\mathbb{R})$ is a core for H_ϵ, it follows from (1.49) that $\kappa(\epsilon,\lambda) = \|(H_\epsilon - i\lambda)^{-1}\| \leq C\left(\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)\right)^{-1}$. We next prove the lower bound in (1.48). Define $m(\epsilon,\lambda) = \inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda)$. By the definition of infimum, for any $\delta > 0, \epsilon > 0, \lambda \in \mathbb{R}$, there exists $v_j \in C_0^\infty(\mathbb{R})$ such that $v_j \not\equiv 0$, $\operatorname{supp} v_j \subset K_j$ and $\|P_{j,\epsilon,\lambda}v_j\| < (m(\epsilon,\lambda)+\delta)\|v_j\|$. By setting $u(x) = 2^{-j/2}v_j(2^{-j}x)$, we find that $\|Qu\| < (m(\epsilon,\lambda)+\delta)\|u\|$. Thus, we have $\kappa(\epsilon,\lambda) > (m(\epsilon,\lambda)+\delta)$. Since $\delta > 0$ is arbitrary, we obtain the lower bound in (1.48). □

1.4.2 Proof of Proposition 1.4.1

We begin the proof of Proposition 1.4.1.

(i) If $\operatorname{dist}(\epsilon\lambda, f(\mathbb{R})) \geq \delta$, then

$$|\Im\langle (H_\epsilon - i\lambda)u, u\rangle| = \left|\left\langle \left(\frac{f}{\epsilon} - \lambda\right)u, u\right\rangle\right| \geq \frac{\delta}{\epsilon}\|u\|^2 \;\; for\ all\ u \in \mathcal{D},$$

and we obtain $\kappa(\epsilon,\lambda) \leq \epsilon/\delta$. Before we prove (ii), for f satisfying the Assumption 1.2.4, we define

$$C_f \stackrel{\text{def}}{=} \sup_{j\in\mathbb{N}} \sup_{x\in K_j} 2^{kj}|f(2^j x)| < +\infty,$$

where $k > 0$ is the parameter that governs the asymptotic behavior of $f(x)$ at infinity as in the Assumption 1.2.4.

(ii) Suppose that $\mathrm{dist}(\epsilon\lambda, \mathrm{cv}(f) \cup \{0\}) \geq \delta$. We also assume that $\epsilon|\lambda| \leq \|f\|_{L^\infty} + \delta$, because otherwise we can use the estimate (i). For any $u \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}\, u \subset K_j$ and $u \not\equiv 0$, we have the lower bound

$$\frac{\|P_{j,\epsilon,\lambda}u\|}{\|u\|} \geq \frac{|\Im\langle P_{j,\epsilon,\lambda}u, u\rangle|}{\|u\|^2} = \frac{\left|\left\langle \left[2^{kj} f\left(2^j \cdot\right) - 2^{kj}\epsilon\lambda\right] u, u\right\rangle\right|}{\epsilon 2^{kj}\|u\|^2} \geq \frac{1}{\epsilon}\left(\epsilon|\lambda| - \frac{C_f}{2^{kj}}\right).$$

Since $\epsilon|\lambda| \geq \delta$, taking large enough $J \in \mathbb{N}$ such that $2^{kJ} \geq 2C_f/\delta$, we find that $C_j(\epsilon, \lambda) \geq \delta/(2\epsilon)$ for all $j \geq J$. Thus, we only consider the case $0 \leq j \leq J$ and the problem is reduced to finding a lower bound on $\|(H_\epsilon - i\lambda)u\|$ when $u \in C_0^\infty(\{x \in \mathbb{R}; |x| < R_\delta\})$, for some $R_\delta > 0$. On a bounded domain, we can neglect the bounded term x^{2m} in H_ϵ and only consider the operator

$$Q = -\partial_x^2 + \frac{i}{\epsilon}\left(f(x) - \epsilon\lambda\right).$$

Thus, our result is same as in the case $m = 1$ [6]. We take a partition of unity $\theta_0(x)^2 + \theta_1(x)^2 \equiv 1$ such that $\theta_0, \theta_1 \in C^\infty(\mathbb{R})$, $\mathrm{supp}\,\theta_0 \subset [-2, 2]$ and $\mathrm{supp}\,\theta_1 \subset (-\infty, -1] \cup [1, +\infty)$. Using these functions, for any $\sigma > 0$, we define a new partition of unity as follows:

$$\chi_0(x)^2 + \chi_1(x)^2 \equiv 1, \ \chi_j(x) = \theta_j\left(\frac{f(x) - \epsilon\lambda}{\epsilon^\sigma}\right), \ j = 0, 1. \tag{1.50}$$

Then, it is obvious that there exists $C > 0$ such that

$$m_1^2 := \sup_{x\in\mathbb{R}} \sum_{j=0}^{1} |\nabla\chi_j(x)|^2 \leq C\epsilon^{-2\sigma}, \ m_2^2 := \sup_{x\in\mathbb{R}} \sum_{j=0}^{1} \left(\Delta\chi_j(x)\right)^2 \leq C\epsilon^{-4\sigma}.$$

Thus, it follows from the localization formula (1.44) that

$$2\|Qu\|^2 + \frac{3C}{\epsilon^{4\sigma}}\|u\|^2 + \frac{8C}{\epsilon^{2\sigma}}\|Qu\|\|u\| \geq \|Q\chi_0 u\|^2 + \|Q\chi_1 u\|^2. \tag{1.51}$$

We first estimate the second term of right hand side (1.51). Since $\mathrm{supp}\,\chi_1$ is contained in the set $\{x \in \mathbb{R}; |f(x) - \epsilon\lambda| \geq \epsilon^\sigma\}$, we have

$$\|\chi_1 u\|\|Q\chi_1 u\| \geq \frac{1}{\epsilon}\left|\langle (f(x) - \epsilon\lambda)\chi_1 u, \chi_1 u\rangle\right| \geq \epsilon^{\sigma-1}\|\chi_1 u\|^2$$

and it follows that

$$\|Q\chi_1 u\| \geq \epsilon^{\sigma-1}\|\chi_1 u\|. \tag{1.52}$$

We next estimate the first term of right hand side (1.51). Under the Assumption 1.2.4, f has only a finite number of critical points. Thus, $f^{-1}(\epsilon\lambda)$ is a finite set. We denote $f^{-1}(\epsilon\lambda) = \{x_1, \cdots, x_n\}$. Since $\mathrm{dist}\,(\epsilon\lambda, \mathrm{cv}(f) \cup \{0\}) \geq \delta > 0$ and $\mathrm{supp}\,\chi_0$ is contained in the set $\{x \in \mathbb{R}; |f(x) - \epsilon\lambda| \leq 2\epsilon^\sigma\}$, $\mathrm{supp}\,\chi_0$ can be decomposed as follows:

$$\mathrm{supp}\,\chi_0 = \sum_{j=1}^{n} I_j \ \textit{with}\ I_j \cap I_k = \emptyset\ (j \neq k),\ |I_j| = \mathcal{O}(\epsilon^\sigma),$$

where I_j is an interval centered at x_j and $|I_j|$ is the length of an interval I_j. In particular, we can decompose $\chi_0 u$ as follows:

$$\chi_0 u = \sum_{j=1}^{n} u_j \ \textit{with}\ \mathrm{supp}\,u_j \cap \mathrm{supp}\,u_k = \emptyset,\ (j \neq k).$$

Inside the interval I_j, the operator Q is well approximated by the operator

$$Q_j = -\partial_x^2 + \frac{i}{\epsilon} f'(x_j)(x - x_j).$$

Indeed, using Taylor's expansion of f around x_j, we have

$$\begin{aligned}
\|Qu_j\|^2 &\geq \frac{1}{2}\|Q_j u_j\|^2 - \frac{1}{\epsilon^2}\|\{f(x) - f(x_j) - f'(x_j)(x - x_j)\}u_j\|^2 \\
&\geq \frac{1}{2}\|Q_j u_j\|^2 - \frac{\|f''\|_{L^\infty}^2}{4\epsilon^2}\|(x - x_j)^2 u_j\|^2 \\
&\geq \frac{1}{2}\|Q_j u_j\|^2 - C\epsilon^{4\sigma-2}\|u_j\|^2.
\end{aligned} \tag{1.53}$$

The operator Q_j is unitary equivalent to the following micro local operator:

$$\widetilde{Q_\gamma} = \gamma^{\frac{2}{3}}(-\partial_x^2 \pm ix),\ \gamma = \frac{|f'(x_j)|}{\epsilon}, \tag{1.54}$$

which satisfies

$$\|\widetilde{Q_\gamma} u\| \geq C\gamma^{\frac{2}{3}}\|u\|. \tag{1.55}$$

Indeed, the operator $P = -\partial_x^2 \pm ix$ satisfies the following inequality:

$$\|Pu\|^2 = \|u''\|^2 + \|xu\|^2 \pm 2\Im\langle u', u\rangle \geq \frac{1}{2}\|u''\|^2 + \|xu\|^2 - \frac{3}{2}\|u\|^2.$$

Thus, we find that P is invertible and P^{-1} is a compact operator which implies (1.55). It follows from (1.53) and (1.55) that

$$\|Q\chi_0 u\|^2 = \sum_{j=1}^{n} \|Qu_j\|^2 \geq C(\epsilon^{-4/3} - \epsilon^{4\sigma-2}) \sum_{j=1}^{n} \|u_j\|^2$$
$$= C(\epsilon^{-4/3} - \epsilon^{4\sigma-2})\|\chi_0 u\|^2.$$

Since $\sigma > 0$ is arbitrary, we take $\sigma > 1/6$ so that $\epsilon^{4\sigma-2} \ll \epsilon^{-4/3}$ for small enough $\epsilon > 0$. Then, we have

$$\|Q\chi_0 u\|^2 \geq C\epsilon^{-4/3}\|\chi_0 u\|^2. \tag{1.56}$$

Thus, it follows from (1.51), (1.52) and (1.56) that

$$\|Qu\|^2 + \frac{1}{\epsilon^{4\sigma}}\|u\|^2 \geq C \min\{\epsilon^{2\sigma-2}, \epsilon^{-4/3}\}\|u\|^2.$$

Finally, taking $\sigma < 1/3$ so that $\epsilon^{-4\sigma} \ll \epsilon^{-4/3} \ll \epsilon^{2\sigma-2}$, we have

$$\|Qu\| \geq C\epsilon^{-2/3}\|u\|,$$

which proves $\kappa(\epsilon, \lambda) \leq C\epsilon^{2/3}$. This completes the proof of (ii).

(iii) The assumption $\lim_{\epsilon\to 0} \epsilon\lambda(\epsilon) = \alpha \in \mathrm{cv}(f)\backslash\{0\}$ implies that $\epsilon|\lambda| \geq \delta$ for some fixed $\delta > 0$ and small enough $\epsilon > 0$. Thus, we can reduce the analysis to a bounded domain as in (ii) and again our result is same as in the case $m = 1$ [6]. Under the Assumption 1.2.4, $f^{-1}(\alpha)$ is a finite set and contains at least one critical point of f. However, in general, the set $f^{-1}(\alpha)$ contains non-critical points. Using a partition of unity, we treat the non-critical points separately and their estimates can be done as in the case (ii). To make the argument simple, we assume that $f^{-1}(\alpha)$ consists of critical points only. We consider two different cases, depending on how fast $\epsilon\lambda(\epsilon)$ converges to α as $\epsilon \to 0$.

(A) We first consider the case:

$$\epsilon^{\sigma_1} \leq |\epsilon\lambda - \alpha| \leq \epsilon^{\sigma_2} \tag{1.57}$$

where $0 \leq \sigma_2 < \sigma_1 < 1/2$ and $3\sigma_2 > 5\sigma_1 - 1$. If $\sigma_2 = 0$, we assume that $\epsilon\lambda \to \alpha$ as $\epsilon \to 0$. We need to prove that

$$\|Qu\| \geq C\epsilon^{-\frac{1}{2}}\|u\| \tag{1.58}$$

where $Q = -\partial_x^2 + \frac{i}{\epsilon}(f(x) - \epsilon\lambda)$ as in (ii). We take $\sigma > 0$ such that

$$\sigma_1 < \sigma < \frac{1}{2}, \ \frac{2\sigma_1}{3} + \frac{1}{6} < \sigma < -\frac{\sigma_1}{6} + \frac{\sigma_2}{2} + \frac{1}{3}. \tag{1.59}$$

We use again a partition of unity $\chi_0(x)^2 + \chi_1(x)^2 \equiv 1$ defined by (1.50). It is obvious that $\operatorname{supp}\chi_0$ and $\operatorname{supp}\chi_1^{'}$ are contained in the set $\{x \in \mathbb{R}; |f(x) - \epsilon\lambda| \le 2\epsilon^\sigma\}$. Thus, by the assumption (1.57), for all $x \in \operatorname{supp}\chi_0 \cup \operatorname{supp}\chi_1^{'}$, we have

$$\frac{\epsilon^{\frac{\sigma_1}{2}}}{C_2} \le \frac{|f(x) - \alpha|^{\frac{1}{2}}}{C_1} \le |f^{'}(x)| \le C_1 |f(x) - \alpha|^{\frac{1}{2}} \le C_2 \epsilon^{\frac{\sigma_2}{2}} \tag{1.60}$$

for some $C_1, C_2 \ge 1$. Since,

$$\chi_j^{'}(x) = \frac{f^{'}(x)}{\epsilon^\sigma}\theta_j^{'}\left(\frac{f(x) - \epsilon\lambda}{\epsilon^\sigma}\right),$$
$$\chi_j^{''}(x) = \frac{f^{''}(x)}{\epsilon^\sigma}\theta_j^{'}\left(\frac{f(x) - \epsilon\lambda}{\epsilon^\sigma}\right) + \left(\frac{f^{'}(x)}{\epsilon^\sigma}\right)^2 \theta_j^{''}\left(\frac{f(x) - \epsilon\lambda}{\epsilon^\sigma}\right), \quad j = 0, 1,$$

there exists $C > 0$ such that

$$m_1^2 = \sup_{x\in\mathbb{R}} \sum_{j=0}^{1} |\nabla\chi_j(x)|^2 \le C\epsilon^{\sigma_2 - 2\sigma},$$
$$m_2^2 = \sup_{x\in\mathbb{R}} \sum_{j=0}^{1} (\Delta\chi_j(x))^2 \le C(\epsilon^{-2\sigma} + \epsilon^{2\sigma_2 - 4\sigma}) \le C\epsilon^{2\sigma_2 - 4\sigma}.$$

Thus, it follows from localization formula (1.44) that

$$\|Qu\|^2 + \epsilon^{2\sigma_2 - 4\sigma}\|u\|^2 \ge C\left(\|Q\chi_0 u\|^2 + \|Q\chi_1 u\|^2\right), \tag{1.61}$$

for some $C > 0$. By (1.52), we have

$$\|Q\chi_1 u\| \ge \epsilon^{\sigma - 1}\|\chi_1 u\|. \tag{1.62}$$

We estimate the first term of right hand side (1.61). We denote $f^{-1}(\epsilon\lambda) = \{x_1, \cdots, x_n\}$ and decompose $\chi_0 u = \sum_{j=1}^n u_j$ as in the case of (ii). We remark that there exists $C > 0$ such that

$$\operatorname{supp} u_j \subset \{x \in \mathbb{R}; |x - x_j| \le C\epsilon^{\sigma - \frac{\sigma_1}{2}}\}. \tag{1.63}$$

Indeed, for $x \in \operatorname{supp} u_j$, we have

$$\frac{\epsilon^{\frac{\sigma_1}{2}}}{C_2}|x-x_j| \le |x-x_j| \inf_{x\in \operatorname{supp} u_j} |f^{'}(x)| \le |f(x)-\epsilon\lambda| \le 2\epsilon^{\sigma},$$

here, we used (1.60). Thus, using estimates (1.53), (1.55), (1.60) and (1.63), we have for small enough $\epsilon > 0$,

$$\begin{aligned}\|Qu_j\|^2 &\ge C\left\{\frac{|f^{'}(x_j)|^{\frac{4}{3}}}{\epsilon^{\frac{4}{3}}}\|u_j\|^2 - \frac{1}{\epsilon^2}\|(x-x_j)^2u_j\|^2\right\}\\ &\ge C\left\{\epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}} - \epsilon^{4\sigma-2\sigma_1-2}\right\}\|u_j\|^2 \ge C\epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}}\|u_j\|^2.\end{aligned}$$

Summing over j, we obtain that

$$\|Q\chi_0 u\|^2 \ge C\epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}}\|\chi_0 u\|^2. \tag{1.64}$$

It follows from (1.61), (1.62) and (1.64) that

$$\|Qu\|^2 + \epsilon^{2\sigma_2-4\sigma}\|u\|^2 \ge C\min\left\{\epsilon^{2\sigma-2}, \epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}}\right\}\|u\|^2,$$

for some $C > 0$. By our choice of $\sigma > 0$ (1.59), for small enough $\epsilon > 0$, we have $\epsilon^{2\sigma_2-4\sigma} \ll \epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}} \ll \epsilon^{2\sigma-2}$ and it follows that

$$\|Qu\|^2 \ge C\epsilon^{\frac{2\sigma_1}{3}-\frac{4}{3}}\|u\|^2 \ge C\epsilon^{-1}\|u\|^2,$$

which proves (1.58).

(B) We next consider the case:

$$|\epsilon\lambda - \alpha| \le \epsilon^{\sigma} \ \textit{for some}\ \frac{1}{3} < \sigma < \frac{1}{2}.$$

Then, as before, for all $x \in \operatorname{supp}\chi_0 \cup \operatorname{supp}\chi_1$, we have $|f^{'}(x)| \le C\epsilon^{\frac{\sigma}{2}}$ and

$$m_1^2 = \sup_{x\in\mathbb{R}}\sum_{j=0}^{1}|\nabla\chi_j(x)|^2 \le C\epsilon^{-\sigma},\ m_2^2 = \sup_{x\in\mathbb{R}}\sum_{j=0}^{1}(\Delta\chi_j(x))^2 \le C\epsilon^{-2\sigma}.$$

Thus, it follows from (1.52) and the localization formula (1.44) that

$$\|Qu\|^2 + \epsilon^{-2\sigma}\|u\|^2 \ge C(\|Q\chi_0 u\|^2 + \epsilon^{2(\sigma-1)}\|\chi_1 u\|^2).$$

Since $\epsilon^{-2\sigma} \ll \epsilon^{-1} \ll \epsilon^{2\sigma-2}$ for small enough $\epsilon > 0$, it suffice to show that $\|Q\chi_0 u\|^2 \ge C\epsilon^{-1}\|\chi_0 u\|^2$. Inside the support of u_j, the operator Q is well approximated by the operator

$$Q_j := -\partial_x^2 + \frac{i}{\epsilon}\left\{\frac{f''(x_j)}{2}(x-x_j)^2 - (\epsilon\lambda - \alpha)\right\}.$$

We note that $|x - x_j| \le C\epsilon^{\frac{\sigma}{2}}$ for some $C > 0$ and all $x \in \operatorname{supp} u_j$. Thus, using Taylor's expansion of f around x_j, it follows that

$$\begin{aligned}\|Qu_j\|^2 &\ge \frac{1}{2}\|Q_j u_j\|^2 - \frac{1}{\epsilon^2}\left\|\left\{f(x) - f(x_j) - \frac{f''(x_j)}{2}(x-x_j)^2\right\}u_j\right\|^2 \\ &\ge \frac{1}{2}\|Q_j u_j\|^2 - \frac{\|f'''\|_{L^\infty}^2}{36\epsilon^2}\|(x-x_j)^3 u_j\|^2 \\ &\ge \frac{1}{2}\|Q_j u_j\|^2 - C\epsilon^{3\sigma-2}\|u_j\|^2. \qquad (1.65)\end{aligned}$$

The operator Q_j is unitary equivalent to the following micro local operator:

$$\gamma^{\frac{1}{2}}(-\partial_x^2 \pm ix^2 - i\mu) \text{ where } \gamma = \frac{|f''(x_j)|}{2\epsilon} \text{ and } \mu = \frac{\epsilon\lambda - \alpha}{\epsilon\gamma^{\frac{1}{2}}}.$$

We use the following fact which will be proved in Lemma 1.6.2 (Appendix).

For any $u \in C_0^\infty(\mathbb{R})$ and sufficiently small $\epsilon > 0$,

$$\|(-\partial_x^2 \pm ix^2 - i\mu)u\| \ge \|u\|.$$

Since $\epsilon^{3\sigma-2} \ll \epsilon^{-1}$, it follows from (1.65) that $\|Qu_j\|^2 \ge C\|u_j\|^2$. Thus, summing over j, we obtain

$$\|Q\chi_0 u\|^2 \ge C\|\chi_0 u\|^2.$$

(C) Take $\sigma_1^0 = \frac{11}{30} \in \left(\frac{1}{3}, \frac{1}{2}\right)$. For any $n \in \mathbb{N}$, we define

$$\sigma_2^n = \sigma_1^{n+1} = \frac{11}{6}\sigma_1^n - \frac{1}{3}.$$

It is obvious that $\sigma_2^n < \sigma_1^n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\sigma_1^n = -\infty$. Let n_0 be the smallest integer such that $\sigma_2^n \le 0$. Then, for any $n \le n_0$,

$$\sigma_1^n < \frac{1}{2},\ \sigma_2^n < \sigma_1^n,\ 3\sigma_2^n < 5\sigma_1^n - 1.$$

Applying the argument (A) to all intervals $[\max\{0, \sigma_2^n\}, \sigma_1^n]$, $n = 0, \cdots, n_0$, we obtain that $\|(H_\epsilon - i\lambda)u\| \ge C\epsilon^{-1/2}\|u\|$ when $\lambda \in \mathbb{R}$ satisfies $|\epsilon\lambda(\epsilon) - \alpha| \ge \epsilon^{\frac{11}{30}}$. Thus,

$$|\epsilon\lambda(\epsilon) - \alpha| \ge \epsilon^{\frac{11}{30}} \Longrightarrow \varlimsup_{\epsilon\to 0}\epsilon^{-\frac{1}{2}}\kappa(\epsilon,\lambda) \le C.$$

On the other hand, applying the argument (B) to $\sigma = \frac{11}{30}$, we obtain that

$$|\epsilon\lambda(\epsilon) - \alpha| \le \epsilon^{\frac{11}{30}} \Longrightarrow \overline{\lim_{\epsilon\to 0}}\, \epsilon^{-\frac{1}{2}}\kappa(\epsilon,\lambda) \le C.$$

This completes the proof of (iii).

(iv) We consider the operator

$$P_{j,\epsilon,0} = -2^{-2j}\partial_x^2 + 2^{2mj}x^{2m} + \frac{i}{\epsilon}f(2^j\epsilon).$$

Then, for any $j \ge 1$ and $u \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} u \subset K_j = \{x \in \mathbb{R}; \frac{3}{8} \le |x| \le 1\}$, we have

$$\|u\|\|P_{j,\epsilon,0}u\| \ge |\Re\langle P_{j,\epsilon,0}u, u\rangle| \ge 2^{2mj}\int_{K_j} |x|^{2m}|u(x)|^2 dx \ge 3^{2m}2^{2(j-3)m}\|u\|^2,$$
$$\|u\|\|P_{j,\epsilon,0}u\| \ge |\Im\langle P_{j,\epsilon,0}u, u\rangle| \ge \frac{1}{\epsilon 2^{kj}}\int_{K_j} 2^{kj}|f(2^jx)||u(x)|^2 dx \ge \frac{m_j}{\epsilon 2^{kj}}\|u\|^2,$$

where $m_j(x) = \inf\{2^{kj}|f(2^jx)|; \frac{3}{8} \le |x| \le 1\}$. From the Assumption 1.2.4, we find that $\lim_{j\to\infty} m_j = 1$. Taking large enough $J \in \mathbb{N}$, we find that

$$C_j(\epsilon,0) \ge C\left(2^{mj} + \frac{1}{\epsilon 2^{kj}}\right) \ge C\epsilon^{-\frac{2m}{k+2m}}, \quad \textit{for all} \quad j \ge J.$$

Since $0 \le j \le J$ corresponds to a bounded spatial domain, we can treated as in (ii) and (iii). Hence, we find that

$$\|H_\epsilon u\| \ge C\epsilon^{-\sigma}\|u\|, \quad \textit{where} \quad \sigma = \begin{cases} 1, & \text{if} \quad 0 \notin f(\mathbb{R}), \\ \frac{2}{3}, & \text{if} \quad 0 \in f(\mathbb{R})\backslash \mathrm{cv}(f), \\ \frac{1}{2}, & \text{if} \quad 0 \in \mathrm{cv}(f). \end{cases}$$

Consequently, we obtain that $\kappa(\epsilon,0) \le C\epsilon^{\min\{\frac{2m}{k+2m},\sigma\}}$.

(v) By virtue of (1.48) Lemma 1.4.3, it suffice to show that

$$C_j(\epsilon,\lambda) \ge C\epsilon^{-\min\left\{\frac{2m}{k+3m+1},\frac{1}{2}\right\}}, \ \textit{for all}\ j \in \mathbb{N},\ 0 < \epsilon \ll 1 \ \textit{and}\ \lambda \in \mathbb{R}. \quad (1.66)$$

As in (ii), (iii), we have $C_j(\epsilon,\lambda) \ge C_J\epsilon^{-1/2}$ for $0 \le j \le J$. Hence, we need only consider the case $j > J$. We take $\widetilde{u} \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\widetilde{u} \subset K_j = \left\{x \in \mathbb{R}; \frac{3}{8} \le |x| \le 1\right\}$, $\|\widetilde{u}\| = 1$ and $\|P_{j,\epsilon,\lambda}\widetilde{u}\| \le 2C_j(\epsilon,\lambda)$. As in (iv), we easily find that

$$\|P_{j,\epsilon,\lambda}\widetilde{u}\| \geq C2^{2mj},\ \|P_{j,\epsilon,\lambda}\widetilde{u}\| \geq \frac{\inf_{x\in K_j}|g_j(x)|}{\epsilon 2^{kj}}, \tag{1.67}$$

where

$$g_j(x) = 2^{kj}f(2^jx) - 2^{kj}\epsilon\lambda.$$

If $2^j \geq \epsilon^{-\frac{1}{k+3m+1}}$, the first inequality of (1.67) implies (1.66). If $2^j < \epsilon^{-\frac{1}{k+3m+1}}$, we integrate by parts and obtain the following relation:

$$\|P_{j,\epsilon,\lambda}\widetilde{u}\|^2 + C2^{2(m-1)j}\|x^{m-1}\widetilde{u}\|^2 = \|Q_{j,\epsilon,\lambda}\widetilde{u}\|^2 + 2^{2(m-1)j+1}\|x^m\partial_x\widetilde{u}\|^2 + 2^{4mj}\|x^{2m}\widetilde{u}\|^2,$$

where $Q_{j,\epsilon,\lambda} = P_{j,\epsilon,\lambda} - 2^{2mj}x^{2m}$. Thus, we have $\|P_{j,\epsilon,\lambda}\widetilde{u}\| \geq \|Q_{j,\epsilon,\lambda}\widetilde{u}\| - C2^{(m-1)j}$. Combining this estimate with (1.67), we obtain

$$2C_j(\epsilon,\lambda) \geq \|P_{j,\epsilon,\lambda}\widetilde{u}\| \geq \frac{C}{3}\left(2^{2mj} + \frac{\inf_{x\in K_j}|g_j(x)|}{\epsilon 2^{kj}} + \|Q_{j,\epsilon,\lambda}\widetilde{u}\| - 2^{(m-1)j}\right). \tag{1.68}$$

As is proved by [6], for any $u \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} u \subset K_j$,

$$\|Q_{j,\epsilon,\lambda}u\| \geq \frac{Ch^{2/3}}{\epsilon 2^{kj}}\|u\|,\ \textit{where}\ h^{2/3} = \epsilon^{1/3}2^{(k-2)j/3} = \mathcal{O}\left(\epsilon^{\frac{m+1}{k+3m+1}}\right).$$

Returning to (1.68), we find that

$$C_j(\epsilon,\lambda) \geq C\left(2^{2mj} + \frac{h^{2/3}}{\epsilon 2^{kj}} - 2^{(m-1)j}\right) \geq C\epsilon^{\frac{-2m}{k+3m+1}},$$

which proves (1.66).

1.4.3 Proof of Theorem 1.2.7

According to (v) in Proposition 1.4.1, it is clear that

$$\Psi(\epsilon) = \left(\sup_{\lambda\in\mathbb{R}}\kappa(\epsilon,\lambda)\right)^{-1} \geq C^{-1}\epsilon^{-\nu(m)}.$$

Since $\Sigma(\epsilon) \geq \Psi(\epsilon)$, we find that $\Sigma(\epsilon) \geq C^{-1}\epsilon^{-\nu(m)}$. Hence, we need only prove the upper bound $\Psi(\epsilon) \leq C\epsilon^{-\nu(m)}$.

We first consider the case $k > m-1$. Fix $0 < \epsilon \ll 1$ and $3/8 < x_0 < 1$. We define $j \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $h > 0$ as follows:

$$2^{j-1} < \epsilon^{-\frac{1}{k+3m+1}} \leq 2^j, \quad h^2 = \epsilon 2^{(k-2)j}, \quad \epsilon\lambda = f(2^j x_0).$$

We take $v \in C_0^\infty(\mathbb{R})$ such that $\|v\| = 1$ and $\operatorname{supp} v \subset [-1, 1]$. We define

$$u_h(x) = \frac{1}{h^{1/3}} v\left(\frac{x - x_0}{h^{2/3}}\right), \quad x \in \mathbb{R}. \tag{1.69}$$

Then, it is obvious that $u_h \in C_0^\infty(\mathbb{R})$, $\|u_h\| = 1$ and $\operatorname{supp} u_h \subset K_j$ for sufficiently small $h > 0$. Recalling that

$$P_{j,\epsilon,\lambda} = \frac{1}{\epsilon 2^{kj}}\left(-h^2\partial_x^2 + h^{2/3}x^{2m} + ig_j(x)\right) \ \textit{where}\ g_j(x) = 2^{kj}f(2^j x) - 2^{kj}\epsilon\lambda,$$

we find that there exists $C > 0$ independent of j, ϵ, λ such that

$$\|P_{j,\epsilon,\lambda} u_h\| \leq C\frac{h^{2/3}}{\epsilon 2^{kj}} = C\epsilon^{-\frac{2m}{k+3m+1}}. \tag{1.70}$$

This implies that $C_j(\epsilon, \lambda) \leq C\epsilon^{-\frac{2m}{k+3m+1}}$ and $\kappa(\epsilon, \lambda) \geq C\epsilon^{\frac{2m}{k+3m+1}}$ by virtue of (1.48) Lemma 1.4.3. Thus, we have

$$\Psi(\epsilon) \leq C\epsilon^{-\frac{2m}{k+3m+1}}. \tag{1.71}$$

It is straightforward to verify (1.70). Using (1.69), we find that $\|h^2\partial_x^2 u_h\| = h^{2/3}\|v''\|$. Since $x^{2m} \leq x_0^{2m} + 2m|x - x_0|$ for all $x \in K_j$, we have $\|x^{2m}u_h\| \leq C$. By our choice of λ, $g_j(x_0) = 0$ and

$$|g_j(x)| \leq |x - x_0| \sup_{\frac{3}{8} \leq |x| \leq 1} |g_j'(x)| \leq C|x - x_0|,$$

where C does not depend on j by the Assumption 1.2.4. Thus, we have $\|g_j u_h\| \leq Ch^{2/3}$ and the proof of (1.70) is complete.

We next consider the case $k \leq m - 1$. Let x_0 be a critical point of f. We assume without loss of generality that $x_0 = 0$. We define

$$\lambda = \frac{f(0)}{\epsilon}, \quad g(x) = f(x) - \epsilon\lambda.$$

We take $v \in C_0^\infty(\mathbb{R})$ such that $\|v\| = 1$ and $\operatorname{supp} v \subset [-1, 1]$ and define

$$u_\epsilon(x) = \frac{1}{\epsilon^{1/8}} v\left(\frac{x}{\epsilon^{1/4}}\right).$$

Using Taylor's expansion of g around $x_0 = 0$, we find that

$$\begin{aligned}\|(H_\epsilon - i\lambda)u_\epsilon\| &\leq \|u_\epsilon^{''}\| + \|x^{2m}u_\epsilon\| + \epsilon^{-1}\|gu_\epsilon\| \\ &= C\epsilon^{-1/2} + C + C\|x^2u_\epsilon\| + \mathcal{O}\left(\int_{\operatorname{supp} u_\epsilon} x^6|u_\epsilon(x)|^2dx\right)^{1/2} \\ &\leq C\epsilon^{-1/2}.\end{aligned}$$

Hence, $C^{-1}\epsilon^{1/2} \leq \sup_{\lambda\in\mathbb{R}} \|(H_\epsilon - i\lambda)^{-1}\|$ and we obtain that

$$\Psi(\epsilon) \leq C\epsilon^{-1/2}. \tag{1.72}$$

Thus, it follows from (1.71) and (1.72) that

$$\Psi(\epsilon) \leq C\epsilon^{-\min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}}.$$

This completes the proof of Theorem 1.2.7.

1.5 Spectral lower bounds-Proof of Theorem 1.2.9.

In this section, we prove Theorem 1.2.9. I. Gallagher, T. Gallay and F. Nier [6] have proved Theorem 1.2.9 for the case $m = 1$, by using a complex deformation method and the same localization techniques as in the proof of Proposition 1.4.1. They also use accurate numerical computations to show that the lower bound in Theorem 1.2.9 is optimal when $m = 1$, in the sense that the exponent $\nu^{'}(m)$ cannot be improved. Our proof for the general case patterns after that of Theorem 1.9 of [6].

1.5.1 Proof of Theorem 1.2.9

To prove Theorem 1.2.9, we use the following Lemma which will be proved later.

Lemma 1.5.1. *There exists $C_0, C_1 > 0$ such that for any $0 < \epsilon \ll 1$,*

$$\sigma(H_\epsilon) \cap \{z \in \mathbb{C}; C_0\Re z \leq |\Im z| \leq \frac{C_1}{\epsilon}\} = \emptyset.$$

We begin the proof of Theorem 1.2.9. We now omit the proof of this lemma and proceed the proof of Theorem 1.2.9. By virtue of Proposition 1.4.1, for any $\lambda \in \mathbb{R}$ such that $|\lambda| \geq C_1\epsilon^{-1}$, we have $\kappa(\epsilon, \lambda) = \|(H_\epsilon - i\lambda)^{-1}\| \leq C\epsilon^{\frac{1}{2}}$ for some $C > 0$. Since

$$\frac{1}{\mathrm{dist}(i\mu, \sigma(H_\epsilon))} \leq \|(H_\epsilon - i\mu)^{-1}\|, \ for\ any\ \mu \in \mathbb{R}, \tag{1.73}$$

it follows that

$$\Re z \geq \frac{C}{\epsilon^{1/2}}, \ for\ any\ z \in \sigma(H_\epsilon) \cap \left\{ z \in \mathbb{C}; |\Im z| \geq \frac{C_1}{\epsilon} \right\}. \tag{1.74}$$

We next consider the domain

$$\sigma(H_\epsilon) \cap \left\{ z \in \mathbb{C}; |\Im z| \leq \frac{C_1}{\epsilon} \right\}$$

By virtue of Lemma 1.5.1, we need only consider the domain

$$\sigma(H_\epsilon) \cap \left\{ z \in \mathbb{C}; |\Im z| \leq \frac{1}{\epsilon^{\nu'(m)}} \right\}, \ where\ \nu'(m) = \min\left\{ \frac{2m}{k+2m}, \frac{1}{2} \right\}.$$

Thus, we take any $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \epsilon^{-\nu'(m)}$, and estimate $\kappa(\epsilon, \lambda) = \|(H_\epsilon - i\lambda)^{-1}\|$ as in the proof of Proposition 1.4.1. By the definition of $C_j(\epsilon, \lambda)$ as in Lemma 1.4.3, there exists $u \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\, u \subset K_j = \{x \in \mathbb{R}; \frac{3}{8} \leq |x| \leq 1\}$, $\|u\| = 1$ and $\|P_{j,\epsilon,\lambda}\| \leq 2C_j(\epsilon, \lambda)$. As in the proof of (iv) of Proposition 1.4.1, we have

$$\begin{aligned} &\|P_{j,\epsilon,\lambda}\| \geq 3^{2m} 2^{2(j-3)m}, \\ &\|P_{j,\epsilon,\lambda}\| \geq \frac{1}{\epsilon 2^{kj}} \inf_{x \in K_j} |g_j(x)|, \ where\ g_j(x) = 2^{kj} f(2^j x) - 2^{kj}\epsilon\lambda. \end{aligned}$$

Hence, we find that for all $j \in \mathbb{N}$

$$C_j(\epsilon, \lambda) \geq \frac{1}{2} \left\{ 3^{2m} 2^{2(j-3)m} + \frac{1}{\epsilon 2^{kj}} \inf_{x \in K_j} |g_j(x)| \right\}.$$

If j is finite, by virtue of Proposition 1.4.1, we have $C_j(\epsilon, \lambda) \geq C\epsilon^{-\frac{1}{2}}$ and it follows that $C_j(\epsilon, \lambda) \geq C\epsilon^{-\nu'(m)}$. Thus, it suffice to consider the case $j \in \mathbb{N}$ is large enough. By the Assumption 1.2.4, if $j \in \mathbb{N}$ is large enough, there exists $C \geq 1$ such that

$$\frac{1}{C|x|^k} \leq 2^{kj} f(2^j x) \leq \frac{C}{|x|^k}, \ for\ all\ x \in K_j = \{x \in \mathbb{R}; \frac{3}{8} \leq |x| \leq 1\}.$$

If $2^{kj}\epsilon|\lambda| \leq \frac{3}{8}$, then $\inf_{x\in K_j}|g_j|(x) \geq \frac{5}{8}$ and we find that

$$C_j(\epsilon,\lambda) \geq C\left\{2^{2mj} + \frac{1}{\epsilon 2^{kj}}\right\} \geq C\epsilon^{-\frac{2m}{k+2m}} \geq C\epsilon^{-\nu'(m)}.$$

On the other hand, if $2^{kj}\epsilon|\lambda| > \frac{3}{8}$ and $|\lambda| \leq \epsilon^{-\nu'(m)}$, then it is obvious that

$$C_j(\epsilon,\lambda) \geq \frac{1}{2}3^{2m}2^{2(j-3)m} \geq C\epsilon^{-\nu'(m)}.$$

Hence, for any $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \epsilon^{-\nu'(m)}$, we have $C_j(\epsilon,\lambda) \geq C\epsilon^{-\nu'(m)}$ for all $j \in \mathbb{N}$, and it follows that

$$\kappa(\epsilon,\lambda)^{-1} \geq C\inf_{j\in\mathbb{N}} C_j(\epsilon,\lambda) \geq C\epsilon^{-\nu'(m)}.$$

Using again (1.73), we find that

$$\Re z \geq \frac{C}{\epsilon^{\nu'(m)}},\ \textit{for all}\ z \in \sigma(H_\epsilon) \cap \left\{z \in \mathbb{C}; |\Im z| \leq \frac{1}{\epsilon^{\nu'(m)}}\right\}. \tag{1.75}$$

Thus, it follows from (1.74) and (1.75) that

$$\Sigma(\epsilon) \geq \frac{C}{\epsilon^{\nu'(m)}},\ \textit{where}\ \nu'(m) = \min\left\{\frac{2m}{k+2m}, \frac{1}{2}\right\}.$$

This completes the proof of Theorem 1.2.9 and it remains to prove Lemma 1.5.1.

1.5.2 Proof of Lemma 1.5.1

To prove Lemma 1.5.1, we use a complex deformation method using the dilation group $(U_\theta\phi)(x) = e^{\theta/2}\phi(e^\theta x)$, which are unitary operators when $\theta \in \mathbb{R}$. If f is given by $f(x) = (1+x^2)^{-k/2}$, the multiplication operator $(i/\epsilon)f(x)$ is a dilation analytic perturbation of $H_\infty = -\partial_x^2 + x^{2m}$. According to the dilation analytic theory ([4]), when we define the operator $H_\epsilon(\theta)$ by

$$H_\epsilon(\theta) = U_\theta H_\epsilon U_\theta^{-1} = -e^{-2\theta}\partial_x^2 + e^{2m\theta}x^{2m} + \frac{i}{\epsilon}\frac{1}{(1+e^{2\theta}x^2)^{k/2}},$$

for $S = \{\theta \in \mathbb{C}; |\Im(\theta)| \leq \pi/4m\}$, the spectrum of $H_\epsilon(\theta)$ does not depend on $\theta \in S$.

Let $\theta = it_k$ and $t_k = \frac{\pi}{4m(k+2)}$. We note that the operator $H_\epsilon(it_k)$ is maximal accretive. Indeed, for any $x, y \in \mathbb{R}$, we have

$$e^{-2it_k}x^2 + e^{2mit_k}y^2 \in \{z \in \mathbb{C}; -2t_k \le \arg z \le 2mt_k\} \subset \{z \in \mathbb{C}; \Re z \ge 0\},$$
$$\frac{i}{\epsilon}\frac{(1+e^{-2it_k}x^2)^{k/2}}{|1+e^{2it_k}x^2|^k} \in \{z \in \mathbb{C}; \frac{\pi}{2} - kt_k \le \arg \le \frac{\pi}{2}\} \subset \{z \in \mathbb{C}; \Re z \ge 0\},$$

and it follows that

$$\Re \langle H_\epsilon(it_k)u, u\rangle \ge 0, \ for\ any\ u \in \mathcal{D}.$$

We take a partition of unity $\chi_0(x)^2 + \chi_1(x)^2 \equiv 1$ such that $\operatorname{supp}\chi_0 \subset (-1,1)$ and $\chi_0(x) = 1$ on $[-1/2, 1/2]$, Then, it is obvious that

$$m_1^2 = \sup_{x\in\mathbb{R}^d}\sum_{j\in J}|\nabla\chi_j(x)|^2 < +\infty, \ m_2^2 = \sup_{x\in\mathbb{R}^d}\sum_{j\in J}(\Delta\chi_j(x))^2 < +\infty,$$

and we apply the localization formula (1.44) to the operator $Q = H_\epsilon(it_k) - i\lambda$. Then, we have

$$\|(H_\epsilon(it_k)-i\lambda)\phi\|^2 + \|\phi\|^2 \ge C\{\|(H_\epsilon(it_k)-i\lambda)\chi_0\phi\|^2 + \|(H_\epsilon(it_k)-i\lambda)\chi_1\phi\|^2\}. \tag{1.76}$$

We estimate the right hand side of (1.76) respectively.

(i) We estimate the first term in the right hand side (1.76). Define $u_0 = \chi_0 u$, then we have

$$\begin{aligned}
&\|u_0\|\|(H_\epsilon(it_k) - i\lambda)u_0\| \\
&\quad \ge |\Im\langle(H_\epsilon(it_k) - i\lambda)u_0, u_0\rangle| \\
&\quad \ge \frac{1}{\epsilon}\Re\langle(f(e^{it_k}x) - \epsilon\lambda)u_0, u_0\rangle - \sin(2t_k)\|u_0^{'}\|^2 \\
&\quad \ge \frac{1}{\epsilon}\inf_{|x|\le 1}\{\Re f(e^{it_k}x) - \epsilon\lambda\}\|u_0\|^2 - \tan(2t_k)\|u_0\|\|(H_\epsilon(it_k) - i\lambda)u_0\|.
\end{aligned}$$

Here, in the third inequality, we used the fact that for any $u \in \mathcal{D}$,

$$\|u^{'}\| \le \frac{1}{\cos(2t_k)}\Re\langle(H_\epsilon(it_k) - i\lambda)u, u\rangle \le \frac{1}{\cos(2t_k)}\|(H_\epsilon(it_k) - i\lambda)u\|\|u\|.$$

Since $|1 + e^{2it_k}| \le 1$ for $|x| \le 1$ and $-kt_k \le \arg f(e^{it_k x}) \le 0$, we have

$$\Re\{f(e^{it_k x})\} \ge |f(e^{it_k x})|\Re(e^{-ikt_k}) \ge 2^{-k/2}\cos(kt_k) \ge 2^{-(k+1)/2}, \quad for\ |x| \le 1.$$

Thus, if we assume that $\epsilon|\lambda| \le l_0 := 2^{-(k+2)/2}$, we have

$$\begin{aligned} 2\|(H_\epsilon(it_k) - i\lambda)u_0\| &\ge (1+\tan(2t_k))\|(H_\epsilon(it_k) - i\lambda)u_0\| \\ &\ge \frac{1}{\epsilon}\inf_{|x|\le 1}\{\Re f(e^{it_k}x) - \epsilon\lambda\}\|u_0\| \\ &\ge \frac{l_1}{\epsilon}\|u_0\| \qquad (1.77) \end{aligned}$$

where $l_1 := 2^{-(k+1)/2} - l_0$.

(ii) We next estimate the second term in the right hand side (1.76). Define $u_1 = \chi_1 u$, then we have

$$\frac{\|(H_\epsilon(it_k) - i\lambda)u_1\|}{\|u_1\|} \ge \left|\frac{\langle H_\epsilon(it_k)u_1, u_1\rangle}{\|u_1\|^2} - i\lambda\right| \ge \inf_{z\in S_k}|z - i\lambda|$$

where S_k is arbitrary sector in the complex plane which contain the quantity

$$\langle H_\epsilon(it_k)u_1, u_1\rangle = e^{-2it_k}\|u_1\|^2 + e^{2mit_k}\|x^m u_1\|^2 + \frac{e^{i(\frac{\pi}{2}-kt_k)}}{\epsilon}\left\langle \frac{(e^{2it_k}+x^2)^{k/2}}{|1+e^{2it_k}x^2|^k}u_1, u_1\right\rangle.$$

We now define the sector S_k as follows:

$$S_k = \{z\in\mathbb{C}; -2t_k \le \arg z \le \frac{\pi}{2} - k(t_k-\delta_k)\},\ \textit{where}\ \delta_k = \frac{1}{2}\arg\left(e^{2it_k} + \frac{1}{4}\right).$$

Then, we find that $\langle H_\epsilon(it_k)u_1, u_1\rangle \in S_k$. Indeed, if we set

$$\begin{aligned} z_1 &= e^{-2it_k}\|u_1\|^2 + e^{2mit_k}\|x^m u_1\|^2, \\ z_2 &= \frac{e^{i(\frac{\pi}{2}-kt_k)}}{\epsilon}\left\langle \frac{(e^{2it_k}+x^2)^{k/2}}{|1+e^{2it_k}x^2|^k}u_1, u_1\right\rangle, \end{aligned}$$

then, it is obvious that $-2t_k < \arg z_1 < 2mt_k$. On the other hand, we have

$$\begin{aligned} &\max_{|x|\ge\frac{1}{2}}\arg\left(e^{2it_k}+x^2\right) = \arg\left(e^{2it_k}+\frac{1}{4}\right) = \delta_k, \\ &\frac{\pi}{2} > \frac{\pi}{2} - k(t_k-\delta_k) > \frac{\pi}{2} - kt_k = \frac{k\pi(2m-1)+4m\pi}{4m(k+2)} > 2mt_k, \end{aligned}$$

it follows that $2mt_k < \arg z_2 < \frac{\pi}{2} - k(t_k - \delta_k)$. Thus, $-2t_k < \arg\langle H_\epsilon(it_k)u_1, u_1\rangle < \frac{\pi}{2} - k(t_k - \delta_k)$ and $\langle H_\epsilon(it_k)u_1, u_1\rangle \in S_k$. Since $\inf_{z\in S_k} |z - i\lambda| = c_k|\lambda|$ for some c_k, we have

$$\|(H_\epsilon(it_k) - i\lambda)u_1\| \geq c_k|\lambda|\|u_1\|. \tag{1.78}$$

Combining (1.76), (1.77) and (1.78), and using the fact that $\|(H_\epsilon(it_k) - i\lambda)\| \geq a_0\cos(2mt_k)$, we obtain that there exists $C_1 > 0$ such that for any $\lambda \in \mathbb{R}$ satisfies $\epsilon|\lambda| \leq l_0$,

$$\|(H_\epsilon(it_k) - i\lambda)u\| \geq C \min\left\{\frac{l_1}{2\epsilon}, c_k|\lambda|\right\} \|u\| \geq \frac{2|\lambda|}{C_1}\|u\|. \tag{1.79}$$

Thus, for any $z = \mu + i\lambda$ with $0 < C_1\mu \leq |\lambda| \leq l_0\epsilon^{-1}$, we have

$$\|(H_\epsilon(it_k) - z)^{-1}\| \leq \frac{\|(H_\epsilon(it_k) - i\lambda)^{-1}\|}{1 - \mu\|(H_\epsilon(it_k) - i\lambda)^{-1}\|} \leq \frac{\frac{C_1}{2|\lambda|}}{1 - \mu\frac{C_1}{2|\lambda|}} \leq \frac{C_1}{|\lambda|},$$

and $z \notin \sigma(H_\epsilon(it_k)) \equiv \sigma(H_\epsilon)$. This completes the proof of Lemma 1.5.1.

1.6 Appendix

In this appendix, we prove Lemma 1.2.2, Lemma 1.2.5, and give Lemma 1.6.1 and Lemma 1.6.2 which were used in the proof of Theorem 1.2.6 and Proposition 1.4.1 (iii) respectively.

1.6.1 Proof of Lemma 1.2.2

Let A be a maximal accretive operator in a Hilbert space X. Suppose that the numerical range $\Theta(A)$ is contained in a sector $S_\alpha = \{z \in \mathbb{C}; |\arg z| \leq \frac{\pi}{2} - 2\alpha\}$ for some $0 < \alpha \leq \pi/4$.

(i) Suppose that $\|e^{-tA}\| \leq Ce^{-\mu t}$ for all $t \geq 0$. Then, by virtue of the representation of the resolvent

$$(A - z)^{-1} = \int_0^\infty e^{-tA}e^{tz}dt, \tag{1.80}$$

we find that the set $\{z \in \mathbb{C}; \Re z < \mu\}$ is contained in a resolvent set $\rho(A)$. Thus, we have $\Sigma \geq \mu$. Setting $z = i\lambda, \lambda \in \mathbb{R}$ in (1.80) and using the fact that e^{-tA} is an infinitesimal C_0-semigroup, we have

$$\|(A-i\lambda)^{-1}\| \le \int_0^\infty \|e^{-tA}\|dt \le \int_0^\infty \min\left\{1, Ce^{-\mu t}\right\} = \frac{\log C + 1}{\mu}.$$

Taking the supremum over $\lambda \in \mathbb{R}$, we thus have

$$\Psi \ge \frac{\mu}{1+\log C}.$$

(ii) We define the line segment in a complex plane as follows:

$$\begin{aligned}
\Gamma_0(\mu,\alpha) &= \left\{z \in \mathbb{C}; \Re z = \mu, |\arg z| \le \frac{\pi}{2} - \alpha\right\}, \\
\Gamma_\pm(\mu,\alpha) &= \left\{z \in \mathbb{C}; \Re z \ge \mu, \arg z = \pm\left(\frac{\pi}{2} - \alpha\right)\right\}, \\
\Gamma(\mu,\alpha) &= \Gamma_-(\mu,\alpha) \cup \Gamma_0(\mu,\alpha) \cup \Gamma_+(\mu,\alpha).
\end{aligned}$$

We use the inverse Laplace formula

$$e^{-tA} = \frac{1}{2\pi i}\int_{\Gamma(\mu,\alpha)} (A-z)^{-1} e^{-tz} dz,$$

where $0 < \mu < \Sigma$. Since $\Re z = \mu$ on the $\Gamma_0(\mu,\alpha)$, we have

$$\left\|\int_{\Gamma_0(\mu,\alpha)} (A-z)^{-1} e^{-zt} dz\right\| \le N(A,\mu)e^{-\mu t}|\Gamma_0(\mu,\alpha)| = N(A,\mu)e^{-\mu t}\frac{2\mu}{\tan 2\alpha}, \tag{1.81}$$

where $N(A,\mu) = \sup_{\Re z=\mu} \|(A-z)^{-1}\|$ and $|\Gamma_0(\mu,\alpha)|$ is the length of a line segment $\Gamma_0(\mu,\alpha)$. On the $\Gamma_+(\mu,\alpha)$, $z \in \Gamma_+(\mu,\alpha)$ can be written as $z = x + (ix/\tan\alpha)$ with $x \ge \mu$. Since $\Theta(A) \subset S_\alpha$, we have

$$\|(A-z)^{-1}\| \le \frac{1}{\operatorname{dist}(z,\Theta(A))} \le \frac{1}{\operatorname{dist}(z,S_\alpha)} = \frac{1}{x}.$$

Thus, we have

$$\begin{aligned}
\left\|\int_{\Gamma_+(\mu,\alpha)} (A-z)^{-1} e^{-tz} dz\right\| &\le \int_\mu^\infty \frac{e^{-tx}}{x}\left(1 + \frac{1}{\tan^2\alpha}\right)^{\frac{1}{2}} dx \\
&\le \frac{1}{\mu t \sin\alpha}\int_{\mu t}^\infty e^{-x} dx = \frac{e^{-\mu t}}{\mu t \sin\alpha}.
\end{aligned} \tag{1.82}$$

Similarly, on the $\Gamma_-(\mu,\alpha)$, we have

$$\left\| \int_{\Gamma_-(\mu,\alpha)} (A-z)^{-1} e^{-tz} dz \right\| \le \frac{e^{-\mu t}}{\mu t \sin\alpha}. \tag{1.83}$$

Using the fact that $\|e^{-tA}\| \le 1 < 1/\sin\alpha$ and combining the estimates (1.81), (1.82) and (1.83), we have

$$\begin{aligned}\|e^{-tA}\| &\le \frac{\mu}{\pi \tan\alpha} N(A,\mu) e^{-\mu t} + \frac{1}{\pi \sin\alpha} \min\left\{\pi, \frac{e^{-\mu t}}{\mu t}\right\} \\ &\le \frac{1}{\pi}\left\{\frac{\mu}{\tan\alpha} N(A,\mu) + \frac{2\pi}{\sin\alpha}\right\} e^{-\mu t}.\end{aligned}$$

(iii) Suppose that $0 < \mu < \Psi$. Since $\mu\|(A-i\lambda)^{-1}\| \le \mu\Psi^{-1} < 1$ for any $\lambda \in \mathbb{R}$, we have,

$$\|(A-\mu-i\lambda)^{-1}\| \le \|\left(I-(A-i\lambda)^{-1}\right)^{-1}\|\|(A-i\lambda)^{-1}\| \le \frac{\Psi^{-1}}{1-\mu\Psi^{-1}} = \frac{1}{\Psi-\mu}.$$

Thus, $N(A,\mu) \le (\Psi-\mu)^{-1}$. This completes the proof of Lemma 1.2.2.

1.6.2 Proof of Lemma 1.2.5

Let $V(x,\epsilon) = x^{2m} + \frac{f'(x)^2}{\epsilon^2}$. By virtue of the Min-Max principle, it suffice to show that there exists $C \ge 1$ such that for any $\phi \in \mathcal{D}$, $0 < \epsilon \ll 1$,

$$\langle \hat{H}_\epsilon \phi, \phi \rangle = \int_{\mathbb{R}} (|\partial_x \phi|^2 + V(x,\epsilon)|\phi|^2) dx \ge \frac{\|\phi\|^2}{C\epsilon^{2\rho(m)}}, \tag{1.84}$$

and

$$\langle \hat{H}_\epsilon \phi_\epsilon, \phi_\epsilon \rangle \le \frac{C}{\epsilon^{2\rho(m)}} \|\phi_\epsilon\|^2 \ for\ some\ \phi_\epsilon \in C_0^\infty(\mathbb{R}) \subset \mathcal{D}. \tag{1.85}$$

We first prove (1.84). Under the Assumption 1.2.4, there exists $L > 0$ such that

$$f'(x)^2 \ge \frac{k^2}{2|x|^{2(k+1)}} \ for\ |x| \ge L,$$

and f has only a finite number of critical points: $\{x_1, \cdots, x_N\}$. We take a partition of unity $\sum_{j=0}^N \chi_j^2 = 1$ such that $\chi_j \in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi_0 \subset (-\infty,-L) \cup (L,+\infty)$ and f has exactly one critical point in $\operatorname{supp}\chi_j$ for $1 \le j \le N$. Applying the IMS localization formula (1.45) to the operator $Q = \hat{H}_\epsilon$, we find that there exists $C > 0$ such that

$$\int_{\mathbb{R}}(|\partial_x\phi|^2+V(x,\epsilon)|\phi|^2)dx+C\|\phi\|^2 \geq \sum_{j=0}^{N}\int_{\mathbb{R}}(|\partial_x\phi_j|^2+V(x,\epsilon)|\phi_j|^2)dx, \quad (1.86)$$

with $\phi_j = \chi_j\phi$. By the mean value theorem, for any $1 \leq j \leq N$, there exists $c_j > 0$ such that for any $x \in \operatorname{supp}\chi_j$,

$$V(x,\epsilon) \geq \frac{f'(x)^2}{\epsilon^2} \geq c_j^2\frac{(x-x_j)^2}{\epsilon^2}.$$

Hence, we have $\langle \hat{H}_\epsilon\phi_j, \phi_j\rangle \geq c_j\epsilon^{-1}\|\phi_j\|^2$ for all $1 \leq j \leq N$. Since

$$V(x,\epsilon) \geq x^{2m} + \frac{k^2}{2\epsilon^2 x^{2(k+1)}} \geq c_0\epsilon^{-\frac{2m}{k+m+1}}$$

for all $x \in \operatorname{supp}\chi_0$, it is obvious that $\langle \hat{H}_\epsilon\phi_0, \phi_0\rangle \geq c_0\epsilon^{-\frac{2m}{k+m+1}}\|\phi_0\|^2$. Thus, we have

$$\langle \hat{H}_\epsilon\phi_j, \phi_j\rangle \geq c_0\epsilon^{-\frac{2m}{k+m+1}}\|\phi_0\|^2 + \epsilon^{-1}\sum_{j=1}^{N} c_j\|\phi_j\|^2, \quad (1.87)$$

and (1.84) follows from (1.86) and (1.87). It remains to prove the upper bound (1.85). We first consider the case $k \leq m-1$. Let x_0 be a critical point of f. We assume without loss of generality that $x_0 = 0$. Take $\phi \in C_0^\infty(-1,1)$ such that $\|\phi\| = 1$ and define $\phi_\epsilon(x) = \epsilon^{-1/4}\phi(\epsilon^{-1/2}x)$. Using Taylor's expansion of f around $x_0 = 0$, we find that

$$\begin{aligned}\langle \hat{H}_\epsilon\phi_\epsilon, \phi_\epsilon\rangle &= \|\phi_\epsilon'\|^2 + \|x^m\phi_\epsilon\|^2 + \epsilon^{-2}\int_{\operatorname{supp}\phi_\epsilon} f'(x)^2|\phi_\epsilon(x)|^2dx \\ &= C\epsilon^{-1} + C + \epsilon^{-2}\mathcal{O}\left(\int_{\operatorname{supp}\phi_\epsilon} x^2|\phi_\epsilon(x)|^2dx\right) \\ &\leq C\epsilon^{-1}. \quad (1.88)\end{aligned}$$

We next consider the case $k > m-1$. Take $\phi \in C_0^\infty(1,2)$ such that $\|\phi\| = 1$ and define $\phi_\epsilon(x) = \epsilon^{\frac{1}{2(k+m+1)}}\phi(\epsilon^{\frac{1}{k+m+1}}x)$. Then, we have

$$\langle \hat{H}_\epsilon\phi_\epsilon, \phi_\epsilon\rangle \leq C\epsilon^{-\frac{2m}{k+m+1}}. \quad (1.89)$$

Thus, (1.85) follows from (1.88) and (1.89).

1.6.3 Proof of Lemma 1.6.1

Lemma 1.6.1. *Let $\beta : \mathbb{R} \to \mathbb{R}_+$ be the function defined by (1.28). Then, there exists $C > 0$ such that for any $\phi \in \mathcal{D}$, $0 < \epsilon \ll 1$,*

$$\int_{\mathbb{R}} \left(|\partial_x \phi|^2 + x^{2m}|\phi|^2 + \frac{\beta(x) f'(x)^2}{\epsilon}|\phi|^2 \right) dx \geq \frac{C}{\epsilon^{\nu(m)}}\|\phi\|^2,\ \nu(m) = \min\left\{ \frac{2m}{k+3m+1}, \frac{1}{2} \right\}. \tag{1.90}$$

Proof. We assume that $A > 0$ is large enough so that all critical points of f are contained in $[-A+1, A+1]$. Then, we have

$$f'(x)^2 \geq \frac{k^2}{2|x|^{2(K+1)}},\ |x| \geq A.$$

Let $W(x,\epsilon) = x^{2m} + \frac{\beta(x)f'(x)^2}{\epsilon}$. Using a same partition of unity as in the proof of Lemma 1.2.5, and applying the IMS localization formula (1.45) to the operator $-\partial_x^2 + V(x,\epsilon)$, there exists $C > 0$ such that

$$\int_{\mathbb{R}} \left(|\partial_x \phi|^2 + W(x,\epsilon)|\phi|^2 \right) dx + C\|\phi\|^2 \geq \sum_{j=0}^{N} \int_{\mathbb{R}} \left(|\partial_x \phi_j|^2 + W(x,\epsilon)|\phi_j|^2 \right) dx \tag{1.91}$$

with $\phi_j = \chi\phi$. By the mean value theorem, for any $1 \leq j \leq N$, there exists $c_j > 0$ such that for any $x \in \operatorname{supp}\chi_j$,

$$W(x,\epsilon) \geq \beta_0 \frac{f'(x)^2}{\epsilon} \geq c_j^2 \frac{(x-x_j)^2}{\epsilon}.$$

Hence, for all $1 \leq j \leq N$, we have

$$\int_{\mathbb{R}} \left(|\partial_x \phi_j|^2 + W(x,\epsilon)|\phi_j|^2 \right) dx \geq c_j \epsilon^{-1/2}\|\phi_j\|^2. \tag{1.92}$$

Since

$$W(x,\epsilon) \geq x^{2m} + \frac{\beta_0 k^2}{2A^{k-m+1}\epsilon|x|^{k+m+1}} \geq C\epsilon^{-\frac{2m}{k+3m+1}},\ for\ A \leq |x| \leq B_\epsilon := A\epsilon^{-\frac{1}{k+3m+1}}$$

and

$$W(x,\epsilon) \geq x^{2m} + \frac{\beta_0 \epsilon^{-\frac{k-m+1}{k+3m+1}} k^2}{2\epsilon|x|^{2(k+1)}} \geq C\epsilon^{-\frac{2m}{k+3m+1}},\ for\ |x| \geq B_\epsilon,$$

it is obvious that

$$\int_{\mathbb{R}} \left(|\partial_x \phi_0|^2 + W(x,\epsilon)|\phi_0|^2\right) dx \geq c_0 \epsilon^{-\frac{2m}{k+3m+1}} \|\phi_0\|^2. \tag{1.93}$$

Thus, (1.90) follows from (1.91), (1.92) and (1.93). □

1.6.4 Proof of Lemma 1.6.2

Lemma 1.6.2. *For any $u \in C_0^\infty(\mathbb{R})$ and sufficiently small $\epsilon > 0$,*

$$\|(-\partial_x^2 \pm ix^2 - i\mu)u\| \geq \|u\|.$$

Proof. We set $P = -\partial_x^2 \pm i(x^2 - \mu)$. We note that if $\epsilon > 0$ is sufficiently small, $|\mu|$ is very large. Hence, we may assume $|\mu| \gg 1$. Since

$$\|Pu\|\|u\| \geq |\Im\langle Pu, u\rangle| \geq \left| \int_{\mathbb{R}} (x^2 - \mu)|u(x)|^2 \right|,$$

if μ is negative, it is obvious that $\|Pu\| \geq |\mu|\|u\| \geq \|u\|$. Thus, we need only consider the case $\mu \gg 1$. We take a partition of unity $\sum_{j=0}^4 \theta_j(x) \equiv 1$ such that $\theta_j \in C^\infty(\mathbb{R})$ and

$$\begin{aligned}
&\operatorname{supp}\theta_0 \subset \left\{x \in \mathbb{R}; -\frac{1}{2} \leq x \leq \frac{1}{2}\right\},\\
&\operatorname{supp}\theta_1 \subset \left\{x \in \mathbb{R}; \frac{1}{4} \leq x \leq 3\right\},\\
&\operatorname{supp}\theta_2 \subset \left\{x \in \mathbb{R}; x \geq 2\right\},\\
&\operatorname{supp}\theta_3 \subset \left\{x \in \mathbb{R}; -3 \leq x \leq -\frac{1}{4}\right\},\\
&\operatorname{supp}\theta_4 \subset \left\{x \in \mathbb{R}; x \leq -2\right\}.
\end{aligned}$$

Using these functions, we define a new partition of unity as follows:

$$\sum_{j=0}^4 \chi_j(x) \equiv 1,\ \chi_j(x) = \theta_j\left(\frac{x}{\mu^{\frac{1}{2}} + \mu^{\frac{1}{4}}}\right),\ j = 0,1,2,3,4.$$

Then, it is obvious that there exists $C > 0$ (μ independent) such that

$$m_1^2 = \sup_{x \in \mathbb{R}^d} \sum_{j=0}^4 |\nabla\chi_j(x)|^2 \leq C\mu^{-\frac{1}{2}},\ m_2^2 = \sup_{x \in \mathbb{R}^d} \sum_{j=0}^4 (\Delta\chi_j(x))^2 \leq C\mu^{-1}.$$

Thus, it follows from the localization formula (1.44) that

$$\|Pu\|^2 + \frac{1}{\mu}\|u\|^2 \geq C\sum_{j=0}^{4}\|P\chi_j u\|^2. \tag{1.94}$$

Set $u_j = \chi_j u$. Since

$$\begin{aligned}
&\operatorname{supp}\chi_0 \subset \left\{x \in \mathbb{R}; |x| \leq \frac{\mu^{\frac{1}{2}} + \mu^{\frac{1}{4}}}{2}\right\},\\
&\operatorname{supp}\chi_1 \subset \left\{x \in \mathbb{R}; \frac{\mu^{\frac{1}{2}} + \mu^{\frac{1}{4}}}{4} \leq x \leq 3(\mu^{\frac{1}{2}} + \mu^{\frac{1}{4}})\right\},\\
&\operatorname{supp}\chi_2 \subset \left\{x \in \mathbb{R}; x \geq 2(\mu^{\frac{1}{2}} + \mu^{\frac{1}{4}})\right\},
\end{aligned}$$

we have

$$\|Pu_0\|\|u_0\| \geq |\Im\langle Pu_0, u_0\rangle| = \int_{\mathbb{R}}(\mu - x^2)|u_0(x)|^2dx \geq \mu\|u_0\|^2, \tag{1.95}$$

$$\|Pu_1\|\|u_1\| \geq \int_{\mu^{\frac{1}{2}}+\frac{\mu^{\frac{1}{2}}+\mu^{\frac{1}{4}}}{4}}^{3(\mu^{\frac{1}{2}}+\mu^{\frac{1}{4}})}(x^2 - \mu)|u_1(x)|^2dx \geq \mu\|u_1\|^2, \tag{1.96}$$

$$\|Pu_2\|\|u_2\| \geq \int_{2(\mu^{\frac{1}{2}}+\mu^{\frac{1}{4}})}^{\infty}(x^2 - \mu)|u_2(x)|^2dx \geq \mu\|u_2\|^2. \tag{1.97}$$

Similarly, we obtain that

$$\|Pu_3\| \geq \mu\|u_3\|,\ \|Pu_4\| \geq \mu\|u_4\|. \tag{1.98}$$

It follows from (1.94), (1.95), (1.96), (1.97) and (1.98) that

$$\|Pu\| \geq \mu\|u\| \geq \|u\|.$$

This completes the proof of Lemma 1.6.2.

□

Chapter 2

Schödinger equations with time-independent strong magnetic fields.

2.1 Introduction

We consider time-dependent Schrödinger equations

$$i\partial_t u = H(t)u(t) \equiv -\nabla^2_{A(t)} u + V(t,x)u, \quad \nabla_{A(t)} = \nabla - iA(t,x) \tag{2.1}$$

in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ of square integrable functions, where $A(t,x) = (A_1(t,x), \ldots, A_d(t,x)) \in \mathbb{R}^d$ and $V(t,x) \in \mathbb{R}$ are respectively magnetic vector and electric scalar potentials. We study the existence and the uniqueness of unitary propagators for Eqn. (2.1).

In accordance with the requirement of quantum mechanics we say that a function $u(t,x)$ of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ is a solution of (2.1) if it satisfies the following properties:

(1) $u(t,\cdot)$ is a continuous function of $t \in \mathbb{R}$ with values in $\mathcal{H}$ and $\|u(t,\cdot)\|_{L^2}$ is independent of $t \in \mathbb{R}$.

(2) $u(t,x)$ satisfies Eqn. (2.1) in the sense of distributions.

Suppose that there exists a dense subspace $\Sigma \subset \mathcal{H}$ such that, for every $s \in \mathbb{R}$ and $\varphi \in \Sigma$, Eqn. (2.1) admits a unique solution $u(t,x)$ which satisfies the initial condition $u(s,x) = \varphi(x)$ and that $u(t,\cdot) \in \Sigma$ for every $t \in \mathbb{R}$. Then the solution operator $\Sigma \ni \varphi \mapsto u(t,\cdot)$ extends to a unitary operator $U(t,s)$

in $\mathcal{H}$ and the two parameter family of operators $\{U(t,s)\colon -\infty < t,s < \infty\}$ satisfies the following properties:

(a) $U(t,s)$ is unitary and $(t,s) \mapsto U(t,s) \in \mathbf{B}(\mathcal{H})$ is strongly continuous.

(b) $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = \mathbf{1}$ for every $-\infty < t,s,r < \infty$.

(c) $U(t,s)\Sigma = \Sigma$ and, for every $\varphi \in \Sigma$, $u(t,x) = (U(t,s)\varphi)(x)$ satisfies Eqn. (2.1) in the sense of distributions.

Definition 2.1.1. *We say a two parameter family of operators* $\{U(t,s)\colon -\infty < t,s < \infty\}$ *is a unitary propagator for* (2.1) *on a dense set* Σ *if it satisfies properties* (a), (b) *and* (c) *above.*

Thus, the existence of a unique unitary propagator on a dense subspace of $\mathcal{H}$ implies that Schrödinger equation (2.1) generates a unique quantum dynamics on $\mathcal{H}$. When A and V are t-independent, it is well known that the existence of a unique unitary propagator on $\mathcal{H}$ is equivalent to the essential selfadjointness of Hamiltonian $-\nabla_A^2 + V$ on $C_0^\infty(\mathbb{R}^d)$. The problem of essential selfadjointness has long and extensively been studied by many authors and it has an extensive literature. We record here following two theorems, Theorem 2.1.2 of Leinfelder and Simader([15]) and Theorem 2.1.3 of Iwatsuka([7]) which are relevant to the present work. We need some notation: $(1+|x|^2)^{1/2} = \langle x \rangle$; $L^p = L^p(\mathbb{R}^d)$, $1 \le p \le \infty$ are Lebesgue spaces and $L^p_{\rm loc} = L^p_{\rm loc}(\mathbb{R}^d)$ are their localizations; $\|u\|_p$ is the norm of L^p, $\|u\| = \|u\|_2$ and (u,v) is the inner product of $u, v \in \mathcal{H}$. A function $W(x)$ is said to be of Stummel class if it satisfies the property that

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<\varepsilon} \frac{|W(y)|^2}{|x-y|^{d-4}} dy = 0, \tag{2.2}$$

where $|x-y|^{4-d}$ should be replaced by $|\log|x-y||$ if $d=4$ and by 1 if $1 \le d \le 3$.

Theorem 2.1.2. *Let* $A \in L^4_{\rm loc}$ *and* $\nabla \cdot A \in L^2_{\rm loc}$. *Let* $V = V_1 + V_2$ *with* $V_1 \in L^2_{\rm loc}$ *and* V_2 *of Stummel class. Suppose that, for a constant* $C_* > 0$,

$$V_1(x) \ge -C_* \langle x \rangle^2. \tag{2.3}$$

Then, $H = -\nabla_A^2 + V$ *is essentially selfadjoint on* $C_0^\infty(\mathbb{R}^d)$.

It can be easily seen that conditions in Theorem 2.1.2 are also necessary as far as smoothness is concerned. However, condition (2.3) on on V at infinity can be substantially relaxed if the magnetic field $B(x) = (B_{jk}(x))$

produced by A,, $B_{jk} = \partial_j A_k - \partial_k A_j$, $\partial_j = \partial/\partial x_j$, grows rapidly at infinity. We define

$$|B(x)| = \Big(\sum_{j<k} |B_{jk}(x)|^2\Big)^{\frac{1}{2}}.$$

Theorem 2.1.3. *Let $\rho(r)$ be a continuous function of $r \geq 0$ such that*

$$\int_0^\infty \rho(r)^{-1} dr = \infty.$$

Suppose that A and V are C^∞ and they satisfy that, for constants C_α,

$$|\partial_x^\alpha B(x)| \leq C_\alpha \rho(|x|)^{|\alpha|}(|B(x)| + 1), \quad |\alpha| = 1, 2; \tag{2.4}$$

$$|B(x)| + V(x) \geq -\rho(|x|)^2. \tag{2.5}$$

Then, $H = -\nabla_A^2 + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

We remark that, by virtue of condition (2.4), magnetic fields which behave too wildly at infinity, e.g. $|B(x)| \geq C \exp(\langle x\rangle^{2+\varepsilon})$ or $|B(x)| = C\cos(e^{\langle x\rangle^{2+\varepsilon}})$ for some $C > 0$ and $\varepsilon > 0$, are *excluded* in Theorem 2.1.3. To the best knowledge of authors, it is unknown whether or not Theorem 2.1.3 remains true without this condition.

We now state main results of this paper. We want to remark beforehand that, by virtue of assumptions on time derivatives, $A(t,x)$ and $V(t,x)$ in following theorems may be considered as perturbations of time frozen potentials $A(t_0,x)$ and $V(t_0,x)$ respectively, t_0 being chosen arbitrarily.

Definition 2.1.4. *$M(\mathbb{R}^d)$ is the space of real valued functions $Q(x)$ of class $C^1(\mathbb{R}^d)$ which satisfy for a positive constant $C > 0$ that*

$$Q(x) \geq C\langle x\rangle \text{ and } |\nabla Q(x)| \leq C\langle x\rangle Q(x). \tag{2.6}$$

For $Q \in M(\mathbb{R}^d)$, $-\Delta + Q(x)^2$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ (see Theorem 2.1.2) and *hereafter L_Q will denote its unique selfadjoint extension.* $L_Q \geq -\Delta + C^2x^2$ and L_Q is positive definite; we have

$$D(L_Q) = \{u \in \mathcal{H}\colon \Delta u,\ Q\nabla u,\ Q^2 u \in \mathcal{H}\}, \tag{2.7}$$

$$C^{-1}\|L_Q u\| \leq \|\Delta u\| + \|Q\nabla u\| + \|Q^2 u\| \leq C\|L_Q u\|, \quad u \in D(L_Q) \tag{2.8}$$

for a constant $C > 0$ (see the proof of Lemma 2.4.1).

For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathbf{B}(\mathcal{X},\mathcal{Y})$ is the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X},\mathcal{X})$. We say $f(t,x)$ is of class $C^\alpha(\mathbb{R}_x^d)$ if it is of class C^α with respect to variables $x \in \mathbb{R}^d$. Multiplication operators by $V(t,\cdot)$, $A(t,\cdot)$ and etc. are denoted by $V(t)$, $A(t)$ and etc. respectively; $\dot{A}(t,x) = \partial_t A(t,x)$ and $\dot{V}(t,x) = \partial_t V(t,x)$ are time derivatives. The letter C denotes various constants whose exact values are not important and they may differ at each occurrence.

First two theorems, Theorems 2.1.5 and 2.1.6, may respectively be thought of as time dependent versions of Theorem 2.1.2 and its form version. I is an interval. Under the assumption of Theorem 2.1.5, operators $H_0(t) = -\nabla^2_{A(t)} + V(t,x) + C(t)\langle x\rangle^2$ and $H(t) = -\nabla^2_{A(t)} + V(t,x)$ are essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ by virtue of Theorem 2.1.2. We denote their selfadjoint extensions again by $H_0(t)$ and $H(t)$.

Theorem 2.1.5. *Suppose A and V satisfy following conditions:*

(1) *$A(t,\cdot) \in L^4_{\mathrm{loc}}$ and $\nabla_x \cdot A(t,\cdot) \in L^2_{\mathrm{loc}}$ for all $t \in I$.*

(2) *$V = V_1 + V_2$ with V_1 and V_2 such that $V_1(t,\cdot) \in L^2_{\mathrm{loc}}$ for $t \in I$ and $V_2(t,\cdot)$ of Stummel class uniformly for $t \in I$. There exist a continuous function $C(t)$ and $Q(x) \in M(\mathbb{R}^d)$ such that*

$$V_1(t,x) + C(t)\langle x\rangle^2 \geq Q(x)^2, \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.9}$$

(3) *For* a.e. *$x \in \mathbb{R}^d$, $A(t,x)$ and $V(t,x)$ are absolutely continuous (AC for short in what follows) with respect to $t \in I$ and multiplication operators in $\mathcal{H}$ by following functions are all L_Q-bounded uniformly for $t \in I$:*

$$\dot{V}(t,x), \quad \nabla_x \cdot \dot{A}(t,x), \quad \dot{A}(t,x)^2, \quad \partial_{x_j}\{(\dot{A}(t,x)^2)\},\ j = 1,\dots,d.$$

Then, following statements are satisfied:

(a) *$H_0(t)$ has t-independent domain $\mathcal{D}$ such that $\mathcal{D} \subset D(H(t))$. We equip $\mathcal{D}$ with the graph norm of $H_0(t_0)$, $t_0 \in I$ being arbitrary.*

(b) *There uniquely exists a unitary propagator $\{U(t,s)\colon t,s \in I\}$ for* (2.1) *on $\mathcal{H}$ with following properties: $U(t,s) \in \mathbf{B}(\mathcal{D})$; for $\varphi \in \mathcal{D}$, $U(t,s)\varphi$ is continuous in Σ with respect to (t,s), of class C^1 in $\mathcal{H}$ and it satisfies*

$$i\partial_t U(t,s)\varphi = H(t)U(t,s)\varphi, \quad i\partial_s U(t,s)\varphi = -U(t,s)H(s)\varphi. \tag{2.10}$$

A remark on condition (2.9) which corresponds to (2.3) of Theorem 2.1.2 is in order since they look differently from each other. As was mentioned above we are considering Eqn. (2.1) when $A(t,x)$ and $V(t,x)$ satisfy conditions of Theorem 2.1.2 for every fixed $t \in \mathbb{R}$, in particular, that

$$V_1(t,x) \geq -C_*(t)\langle x\rangle^2 \tag{2.11}$$

for a continuous $C_*(t)$. Then, if we choose $C(t) = C_*(t)+C$, $V_1(t,x)$ satisfies (2.9) with $Q(x)^2 = C\langle x\rangle^2 \in M(\mathbb{R}^d)$, C being an arbitrarily large constant. However, this is the *worst* case conceivable and $V_1(t,x)$ may rapidly grow to positive infinity as $|x| \to \infty$, in which case $V_1(t,x)$ certainly satisfies (2.11). If $V_1(t,x)$ increases the faster as $|x| \to \infty$, then $Q(x)$ of (2.9) may be taken the larger, condition (3) becomes the less restrictive and the class of potentials accommodated by the theorem becomes the wider. Condition (2.9) is formulated for studying these cases simultaneously. Similar remark applies to conditions (2.13), (2.23) and (2.24) in following theorems.

When V is spatially more singular than in Theorem 2.1.5, we use quadratic form formalism. The following is a form version of Theorem 2.1.5. A function $W(t,x)$ is said to be of Kato class uniformly for $t \in I$, if

$$\lim_{\varepsilon\to 0} \sup_{t\in I, x\in\mathbb{R}^d} \int_{|x-y|<\varepsilon} \frac{|W(t,y)|}{|x-y|^{d-2}} dy = 0, \tag{2.12}$$

where $|x-y|^{2-d}$ should be replaced by $|\log|x-y||$ if $d=2$ and by 1 if $d=1$. We write $q(u,u) = q(u)$ for quadratic forms $q(u,v)$.

Theorem 2.1.6. *Suppose that A and V satisfy following conditions:*

(1) *$A(t,\cdot) \in L^2_{\rm loc}$ for every $t \in I$.*

(2) *$V(t,x) = V_1(t,x) + V_2(t,x)$ with V_1 such that $V_1(t,\cdot) \in L^1_{\rm loc}(\mathbb{R}^d_x)$ for all $t \in I$ and $V_2(t,\cdot)$ of Kato class uniformly for $t \in I$. There exist a continuous function $C(t)$ and $Q \in M(\mathbb{R}^d)$ such that*

$$V_1(t,x) + C(t)\langle x\rangle^2 \geq Q(x)^2, \quad t \in I. \tag{2.13}$$

(3) *A and V are AC with respect to t for* a.e. *$x \in \mathbb{R}^d$ and*

$$\|\dot{A}(t)L_Q^{-1/2}\|_{\mathbf{B}(L^2)} + \|L_Q^{-1/2}\dot{V}(t)L_Q^{-1/2}\|_{\mathbf{B}(L^2)} \leq C, \quad t \in I \tag{2.14}$$

for a constant $C > 0$.

Then, following statements are satisfied:

(a) *The quadratic form $q_0(t)$ defined on $C_0^\infty(\mathbb{R}^d)$ by*

$$q_0(t)(u) = \int_{\mathbb{R}^d} (|\nabla_{A(t)}u|^2 + (V(t,x) + C(t)\langle x\rangle^2)|u|^2)dx$$

is strictly positive and closable; the closure $[q_0(t)]$ *has domain* $\mathcal{Y}$ *independent of* $t \in I$ *and* $\mathcal{Y} \subset D(L_Q^{\frac{1}{2}})$*. We equip* $\mathcal{Y}$ *with the inner product* $[q_0(t_0)](u, v)$ *by choosing* t_0 *arbitrarily and denote by* $\mathcal{X}$ *its dual space with respect to the inner product of* $\mathcal{H}$*. We have* $H(t) = -\nabla_{A(t)}^2 + V(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ *and* $t \to H(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ *is norm continuous.*

(b) *There uniquely exists a unitary propagator for* (2.1) *on* $\mathcal{Y}$ *with following properties:* $U(t, s) \in \mathbf{B}(\mathcal{Y})$*; for* $\varphi \in \mathcal{Y}$*,* $U(t, s)\varphi$ *is continuous in* $\mathcal{Y}$ *with respect to* (t, s)*, of class* C^1 *in* $\mathcal{X}$ *and satisfies equations* (2.10)*.*

Before stating time dependent versions of Theorem 2.1.3, we generalize it for $V(x)$ which are locally as singular as those in Theorem 2.1.2 or in Theorem 2.1.6 by slightly strengthening conditions (2.4) and (2.5) at infinity.

Theorem 2.1.7. *Let* A *be of class* C^3 *and the magnetic field* B *generated by* A *satisfy for constants* C_α *that*

$$|\partial_x^\alpha B(x)| \le C_\alpha \langle x\rangle^{|\alpha|}(|B(x)| + 1), \quad |\alpha| = 1, 2. \tag{2.15}$$

Let $V(x) = V_1(x) + V_2(x)$ *with* $V_1 \in L^2_{\text{loc}}$ *and* V_2 *of Stummel class. Suppose that there exist constants* $\theta < 1$ *and* $C_* > 0$ *such that*

$$\theta|B(x)| + V_1(x) \ge -C_*\langle x\rangle^2, \quad x \in \mathbb{R}^d. \tag{2.16}$$

Then, $L = -\nabla_A^2 + V$ *is essentially selfadjoint on* $C_0^\infty(\mathbb{R}^d)$ *and the domain of its selfadjoint extension* H *is given by* $D(H) = \{u \in \mathcal{H} \colon -\nabla_A^2 u + Vu \in \mathcal{H}\}$*.*

Theorem 2.1.8. *Let* $A(x)$ *and* $B(x)$ *be as in* Theorem 2.1.7*. Let* $V(x) = V_1(x) + V_2(x)$ *with* $V_1 \in L^1_{\text{loc}}(\mathbb{R}^d_x)$ *and* V_2 *of Kato class. Suppose that there exist constants* $\theta < 1$ *and* C_* *such that* (2.16) *is satisfied. Define*

$$\tilde{V}_1(x) = V_1(x) + (C_* + C_1)\langle x\rangle^2 \tag{2.17}$$

with a sufficiently large constant C_1*. Then, following statements are satisfied:*

(1) *The quadratic form* q_0 *on* $C_0^\infty(\mathbb{R}^d)$ *defined by*

$$q_0(u) = \|\nabla_A u\|^2 + ((\tilde{V}_1 + V_2)u, u) \tag{2.18}$$

is bounded from below and closable. The closure has domain

$$D([q_0]) = \{u \in L^2 \colon \nabla_A u \in L^2, \ \ (|B| + |\tilde{V}_1| + \langle x\rangle^2)^{1/2} u \in L^2\}. \tag{2.19}$$

For $u \in D([q_0])$*, we have* $V_2|u|^2 \in L^1$ *and* $[q_0](u)$ *is given by* (2.18)*.*

(2) *The selfadjoint operator* H_0 *defined by* $[q_0]$ *is given by*

$$H_0 u = -\nabla_A^2 u + (\tilde{V}_1 + V_2)u, \tag{2.20}$$

$$D(H_0) = \{u \in D([q_0]),\ -\nabla_A^2 u + (\tilde{V}_1 + V_2)u \in L^2\}. \tag{2.21}$$

Suppose that A and V satisfy conditions of Theorem 2.1.7, then they also satisfy those of Theorem 2.1.8, and the operator H_0 defined in Theorem 2.1.8 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and $D(H_0) = \{u \in L^2\colon (-\nabla_A^2 + \tilde{V}_1 + V_2)u \in L^2\}$. This follows from the fact that selfadjoint operators admit no proper selfadjoint extensions.

Theorems 2.1.9 and 2.1.10 in what follows are time dependent versions of Theorems 2.1.7 and 2.1.8 respectively. Under assumptions of Theorem 2.1.9

$$H(t) = -\nabla_{A(t)}^2 + V(t,x) \quad \text{and} \quad H_0(t) = -\nabla_{A(t)}^2 + V(t,x) + (C(t) + C_1)\langle x\rangle^2$$

are essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ by virtue of Theorem 2.1.7. We denote their selfadjoint extensions again by $H(t)$ and $H_0(t)$.

Theorem 2.1.9. *Suppose that* A *and* V *satisfy following conditions:*

(1) $A(t,x) \in C^3(\mathbb{R}_x^d)$ *for all* $t \in I$ *and the magnetic field* $B(t,x)$ *generated by* $A(t,x)$ *satisfies, for constants* $C_\alpha > 0$,

$$|\partial_x^\alpha B(t,x)| \le C_\alpha \langle x\rangle^{|\alpha|}\langle B(t,x)\rangle, \quad |\alpha| = 1,2, \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.22}$$

(2) $V(t,x) = V_1(t,x) + V_2(t,x)$ *with* $V_1(t,\cdot) \in L^2_{\text{loc}}(\mathbb{R}_x^d)$ *for all* $t \in I$ *and* $V_2(t,\cdot)$ *of Stummel class uniformly with respect to* $t \in I$. *There exist a constant* $\theta < 1$, *a continuous function* $C(t)$ *and* $Q \in M(\mathbb{R}^d)$ *such that*

$$\theta|B(t,x)| + V_1(t,x) + C(t)\langle x\rangle^2 \ge Q(x)^2, \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.23}$$

(3) *For* a.e. $x \in \mathbb{R}^d$, $A(t,x)$ *and* $V(t,x)$ *are* AC *with respect to* $t \in I$. *Time derivatives satisfy, for a constant* $C > 0$, *that*

$$|\nabla_x \cdot \dot{A}(t,x)| + |\dot{A}(t,x)|^2 + |\nabla_x(\dot{A}(t,x)^2)| \le CQ(x)^2, \quad (t,x) \in I \times \mathbb{R}^d;$$

and that $\dot{V}(t,x) = W_0(t,x) + W_1(t,x) + W_2(t,x)$ *such that*

$$\|Q^{-2+j}W_j(t)(-\Delta+1)^{-j/2}\|_{\mathbf{B}(\mathcal{H})} \le C, \quad t \in I, \quad j = 0,1,2.$$

Then, following statements are satisfied for a sufficiently large $C_1 > 0$*:*

(a) *Domain* $\mathcal{D}$ *of* $H_0(t)$ *is independent of* $t \in I$ *and* $\mathcal{D} \subset D(H(t))$ *for all* $t \in I$*. Equip* $\mathcal{D}$ *with the graph norm of* $H_0(t_0)$*,* t_0 *being arbitrarily.*

(b) *There uniquely exists a unitary propagator* $\{U(t,s)\colon t,s \in I\}$ *on* $\mathcal{H}$ *for* (2.1) *such that* $U(t,s) \in \mathbf{B}(\mathcal{D})$*; for* $\varphi \in \mathcal{D}$*,* $U(t,s)\varphi$ *is continuous with respect to* (t,s) *in* $\mathcal{D}$*, of class* C^1 *in* $\mathcal{H}$ *and satisfies* (2.10)*.*

Theorem 2.1.10. *Let* $A(t,x)$ *and* $B(t,x)$ *be as in* Theorem 2.1.9*. Suppose*

(1) $V(t,x) = V_1(t,x) + V_2(t,x)$ *with* $V_1(t,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^d_x)$ *for all* $t \in I$ *and* $V_2(t,\cdot)$ *of Kato class uniformly with respect to* $t \in I$*. There exist a* $\theta < 1$*, a continuous function* $C(t)$ *and* $Q \in M(\mathbb{R}^d)$ *such that*

$$\theta|B(t,x)| + V_1(t,x) + C(t)\langle x\rangle^2 \geq Q(x)^2, \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.24}$$

(2) $V(t,x)$ *is AC with respect to* $t \in I$ *for* a.e. $x \in \mathbb{R}^d$ *and* $\dot V(t,x)$ *satisfies, for a constant* $C > 0$*,*

$$\|L_Q^{-1/2}|\dot V(t)|L_Q^{-1/2}\|_{\mathbf{B}(L^2)} \leq C, \quad t \in I. \tag{2.25}$$

Let $\tilde V = V + (C(t)+C_1)\langle x\rangle^2$ *and* $\tilde V_1 = V_1 + (C(t)+C_1)\langle x\rangle^2$ *for a sufficiently large constant* $C_1 > 0$*. Then, following statements are satisfied.*

(a) *The quadratic form* $q_0(t)$ *on* $C_0^\infty(\mathbb{R}^d)$ *defined by*

$$q_0(t)(u) = \|\nabla_{A(t)}u\|^2 + (\tilde V(t,x)u,u) \tag{2.26}$$

is bounded from below and closable. Domain $\mathcal{Y}$ *of its closure* $[q_0(t)]$ *is given by* (2.19) *with obvious changes.* $\mathcal{Y}$ *is independent of* t *and satisfies* $\mathcal{Y} \subset D(L_Q^{\frac{1}{2}})$*. We equip* $\mathcal{Y}$ *with the inner product* $[q_0(t_0)](u,v)$*,* $t_0 \in I$ *being arbitrarily and denote by* $\mathcal{X}$ *its dual space with respect to the inner product of* $\mathcal{H}$*. For* $t \in I$*, define operator* $H(t)$ *from* $\mathcal{Y}$ *to* $\mathcal{X}$ *by*

$$(H(t)u,v) = (\nabla_{A(t)}u, \nabla_{A(t)}v) + (V(t,x)u,v), \quad u,v \in \mathcal{Y}.$$

Then, $H(t) \in \mathbf{B}(\mathcal{Y},\mathcal{X})$ *and it is norm continuous with respect to* $t \in I$*.*

(b) *There uniquely exists a unitary propagator for* (2.1) *on* $\mathcal{Y}$ *such that* $U(t,s) \in \mathbf{B}(\mathcal{Y})$*; for* $\varphi \in \mathcal{Y}$*,* $U(t,s)\varphi$ *is continuous with respect to* (t,s) *in* $\mathcal{Y}$*, of class* C^1 *in* $\mathcal{X}$ *and satisfies* (2.10)*. Moreover,* $\{U(t,s)\}$ *extends to a strongly continuous family of bounded operators in* $\mathcal{X}$*.*

We emphasize that in all theorems above no conditions are imposed on the behavior at infinity of the positive part of V in contrast to strong size restrictions on its negative part.

For the reference on the problem, we refer to the introduction of [28] and we shall jump into the proof of Theorems immediately. We shall not prove Theorems 2.1.5 and 2.1.6 because they are proved in [28] for the case $Q(x) = C\langle x\rangle$ and the proof goes through for the present cases with obvious changes, and because the proof of Theorems 2.1.9 and 2.1.10 which we shall be devoted to in what follows basically patterns after that of [28], though several new estimates are necessary.

The plan of paper is as follows. Section 2.2 collects some well known results which are necessary in subsequent sections. We prove selfadjointness theorems, Theorems 2.1.7 and 2.1.8 in Section 2.3. In Section 2.4, we formulate and prove an estimate for the resolvent of $H_1(t) = -\nabla^2_{A(t)} + V_1(t,x) + (C(t) + C_1)\langle x\rangle^2$ which replaces the diamagnetic inequality (cf. [5]). We emphasize that it is hopeless to have standard diamagnetic inequality for this operator since the scalar potential $W(t,x) = V_1(t,x) + (C(t) + C_1)\langle x\rangle^2$ of $H_1(t)$ can wildly diverge to negative infinity as $|x| \to \infty$ and $-\Delta + W(t,x)$ is *not* in general essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. We prove Theorems 2.1.9 and 2.1.10 in Section 2.5 and 2.6 respectively by using materials prepared in preceding sections.

2.2 Preliminaries

In this section, we recall Kato's abstract theory of evolution equations which the proof of Theorems will eventually relies upon, and Iwatsuka's identity which will be used for deriving various estimates necessary for applying Kato's theory.

2.2.1 Kato's abstract theory for evolution equations

As in the previous paper [28], Theorems 2.1.9 and 2.1.10 will be proven by applying the following abstract theorem. The theorem is the consequence of Theorem 5.2, Remarks 5.3 and 5.4 of Kato's seminal paper [9].

Theorem 2.2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be a pair of Hilbert spaces such that $\mathcal{Y} \subset \mathcal{X}$ continuously and densely. Let $\{A(t), t \in I\}$, I being an interval, be a family of closed operators in $\mathcal{X}$ with dense domain $D(A(t))$ such that $\mathcal{Y} \subset D(A(t))$ for every $t \in I$ and $I \ni t \to A(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ is norm continuous. Suppose that following conditions are satisfied:*

(1) *For every* $t \in I$, *there exist inner products* $(\cdot,\cdot)_{\mathcal{X}_t}$ *and* $(\cdot,\cdot)_{\mathcal{Y}_t}$ *of* $\mathcal{X}$ *and* $\mathcal{Y}$ *respectively which define norms equivalent to the original ones and which satisfy, for a constant* $c > 0$,

$$\|u\|_{\mathcal{Y}_t}/\|u\|_{\mathcal{Y}_s} \le e^{c|t-s|}, \quad \|u\|_{\mathcal{X}_t}/\|u\|_{\mathcal{X}_s} \le e^{c|t-s|}, \quad u \neq 0. \tag{2.27}$$

(2) *If we let* $\mathcal{X}_t$ *and* $\mathcal{Y}_t$ *be Hilbert spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *with these inner products,* $A(t)$ *is selfadjoint in* $\mathcal{X}_t$ *and the part* $\tilde{A}(t)$ *of* $A(t)$ *in* $\mathcal{Y}_t$ *is also selfadjoint in* $\mathcal{Y}_t$.

Then, there uniquely exists a strongly continuous family of bounded operators $\{U(t,s)\colon t,s \in I\}$ *in* $\mathcal{X}$ *that satisfies*

(a) $U(t,r) = U(t,s)U(s,r)$, $U(s,s) = I$ *for every* t, s *and* $r \in I$.

(b) $U(t,s) \in \mathbf{B}(\mathcal{Y})$; *for* $\varphi \in \mathcal{Y}$, $U(t,s)\varphi$ *is continuous with respect to* (t,s) *in* $\mathcal{Y}$, *of class* C^1 *in* $\mathcal{X}$ *and it satisfies*

$$\partial_t U(t,s)\varphi = -iA(t)U(t,s)\varphi, \quad \partial_s U(t,s)\varphi = iU(t,s)A(s)\varphi. \tag{2.28}$$

2.2.2 Iwatsuka's Identity

In [7], Iwatsuka has found an ingenious formula which rewrites Schrödinger operator $H = -\nabla_A^2 + V$ in the form of elliptic operators in which the magnetic field $B_{jk} = \partial_j A_k - \partial_k A_j$ appears explicitly, which he has used for proving Theorem 2.1.3. We recall it here as we shall use it several times for deriving various estimates. For the proof of following lemmas we refer to Iwatsuka's paper [7], formula (2.12) and proofs of Theorem 1.1 and Theorem 2.1 therein. We denote $b \cdot a = {}^t b a$ for a vector b and a matrix a.

Lemma 2.2.2. *Let* $G(x) = \{G_{jk}\}$ *be Hermitian matrix valued function and*

$$G_{jk} = \alpha_{jk} + i\beta_{jk}, \text{ for real valued } \alpha_{jk} = \alpha_{kj} \text{ and } \beta_{jk} = -\beta_{kj},\ j,k = 1,\dots,d;$$

$F(x) = \{F_j\}$ *be complex vector field such that with real* A *and complex* b

$$F(x) = A(x) + b(x) \tag{2.29}$$

and $B(x) = \{B_{jk}\}$, $B_{jk} = \partial_j A_k - \partial_k A_j$. *Then, we have the following identity:*

$$\begin{aligned} -\nabla_{\overline{F}} \cdot G \nabla_F = -\nabla_A \cdot \alpha \nabla_A + i\{2\Re(\overline{b} \cdot G) - (\nabla \cdot \beta)\}\nabla_A \\ - \sum_{j<k} \beta_{jk} B_{jk} + i\nabla \cdot (Gb) + \overline{b} \cdot Gb. \end{aligned} \tag{2.30}$$

In particular, if $\alpha_{jk} = \delta_{jk}$*, Kronecker's delta and*

$$G_{jk} = \delta_{jk} + i\beta_{jk} \quad \textit{and} \quad b = \tfrac{1}{2}\nabla \cdot \beta \tag{2.31}$$

for a real valued skew-symmetric matrix $\{\beta_{jk}\}$*, then*

$$-\nabla_A^2 = -\nabla_{\overline{F}} \cdot G\nabla_F + \sum_{j<k} \beta_{jk} B_{jk} + R, \tag{2.32}$$

$$R = \tfrac{1}{2} \sum_{j,k} \beta_{jk}\partial_j b_k + \tfrac{1}{4}b^2. \tag{2.33}$$

Real skew-symmetric β in (2.31) is completely arbitrary for identity (2.32) and Iwatsuka's choice in [7] is as follows: Take $\chi \in C^\infty([0,\infty))$ such that $\chi(r) = 1$ for $0 \le r \le 1/2$, $\chi(r) = r^{-1}$ for $r \ge 1$ and

$$0 < r\chi(r) \le 1 \text{ for all } r > 0$$

and define

$$\beta(x) = \chi(|B(x)|)B(x). \tag{2.34}$$

In what follows, $\beta(x)$ *always denotes the function defined by* (2.34) *and* $b(x)$ *and* $R(x)$ *are respectively defined by* (2.31) *and* (2.33) *by using this* $\beta(x)$. We write

$$|\partial B| = \sum_{|\alpha|=1, j<k} |\partial^\alpha B_{jk}| \quad \text{and} \quad |\partial^2 B| = \sum_{|\alpha|=2, j<k} |\partial^\alpha B_{jk}|.$$

Lemma 2.2.3. *Suppose* $A(x)$ *and* $B(x)$ *satisfy* (2.15)*. Then:*

$$|\beta(x)| \le 1, \quad \sum_{j<k} \beta_{jk} B_{jk} = \chi(|B|)|B|^2 \ge |B| - 1, \tag{2.35}$$

$$|\partial_x^\alpha \beta| \le C\langle x\rangle^{|\alpha|}, \quad |\alpha| = 1, 2; \quad |b| \le C\langle x\rangle, \quad |R| \le C\langle x\rangle^2. \tag{2.36}$$

For real skew-symmetric $\tilde\beta = (\tilde\beta_{jk})$, we have (Proposition 4.1 of [7]) that

$$-|\tilde\beta| \le i\tilde\beta \le |\tilde\beta|, \quad |\tilde\beta| = \Big(\sum_{j<k} \tilde\beta_{jk}^2\Big)^{\frac{1}{2}} \tag{2.37}$$

in the sense of quadratic forms on $\mathbf{C}^d$. In what follows we shall use identity (2.32) by modifying $\beta(x)$ of (2.34) in various ways.

2.3 Selfadjointness

We prove Theorems 2.1.7 and 2.1.8 in this section. We take and fix $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^d$,

$$\varphi(x) = 1 \text{ for } |x| \le 1 \text{ and } \varphi(x) = 0 \text{ for } |x| \ge 2. \tag{2.38}$$

We set $\varphi_n(x) = \varphi(x/n)$ for $n = 1, 2, \ldots$ and define for $0 < \theta \le 1$

$$\beta_{n,\theta}(x) = \theta\varphi_n(x)\beta(x). \tag{2.39}$$

The following lemma is obvious by virtue of (2.37).

Lemma 2.3.1. *If we change β by $\beta_{n,\theta}(x)$, then* (2.32) *remains to hold with G, b and R being replaced by corresponding $G_{n,\theta}$, $b_{n,\theta}$, $R_{n,\theta}$. Matrix $G_{n,\theta}$ satisfies*

$$G_{n,\theta}(x) = \mathbf{1} + i\theta\varphi_n(x)\beta(x) \ge \mathbf{1} - \theta, \quad x \in \mathbb{R}^d; \tag{2.40}$$

and $b_{n,\theta}$ and $R_{n,\theta}$ satisfy corresponding estimates in (2.36) *uniformly with respect to θ and n.*

Proof of Theorem 2.1.7 The following is a modification of Kato's argument ([13]). It suffices to show that the image of $L \pm i$, $R(L \pm i)$, is dense in $\mathcal{H}$. Thus we suppose that $f \in \mathcal{H}$ satisfies $f \perp R(L \pm i)$ and show $f = 0$ then. We prove the + case only. The proof for the other case is similar.

We first assume $V_2 = 0$. Define, for $n = 1, 2, \ldots$, $V_n(x) = \chi_{B_{2n}(0)}(x)V(x)$, where $B_{2n}(0) = \{x \in \mathbb{R}^d \colon |x| < 2n\}$ and χ_F is the characteristic function of the set F, and

$$L_n = -\nabla_A^2 + V_n, \quad D(L_n) = C_0^\infty(\mathbb{R}^d).$$

Since $V_n(x)$ is bounded from below, L_n is essentially selfadjoint by virtue of Theorem 2.1.2. It follows that there exists $u_n \in C_0^\infty(\mathbb{R}^d)$ such that

$$\|(L_n + i)u_n - f\| \le 1/n, \quad n = 1, 2, \ldots. \tag{2.41}$$

Then, $\|(L_n + i)u_n\| \le \|f\| + 1/n$ and

$$\|u_n\| \le \|(L_n + i)u_n\| \le C, \quad \|L_n u_n\| \le \|f\| + \|u_n\| + 1/n \le C. \tag{2.42}$$

Let $\varphi_n(x)$ be as above. Then, $\varphi_n(x)V_n(x) = \varphi_n(x)V(x)$ and

$$\varphi_n(x)(L_n + i)u_n = (L + i)\varphi_n u_n + 2(\nabla\varphi_n)\nabla_A u_n + (\Delta\varphi_n)u_n.$$

It follows from (2.41) that

$$\|f\|^2 = \lim_{n\to\infty}(\varphi_n f, (L_n + i)u_n) = \lim_{n\to\infty}(f, \varphi_n(L_n + i)u_n)$$

$$= \lim_{n\to\infty}\{(f,(L+i)\varphi_n u_n) + 2(f,(\nabla\varphi_n)\nabla_A u_n) + (f,(\Delta\varphi_n)u_n)\}. \quad (2.43)$$

The first term on the right vanishes by the assumption and the third satisfies

$$|(f,(\Delta\varphi_n)u_n)| \le n^{-2}\|\Delta\varphi\|_\infty\|f\|\|u_n\| \to 0 \quad (n\to\infty).$$

For estimating $\|\nabla_A u_n\|$, we use Iwatsuka's identity (2.32) with $\beta_{2n,\theta}$ defined by (2.39) with $2n$ replacing n, which produces

$$L_n = -\nabla_A^2 + V_n = -\nabla_{\overline{F_{2n,\theta}}} G_{2n,\theta}\nabla_{F_{2n,\theta}} + W_{2n,\theta}, \quad (2.44)$$

$$F_{2n,\theta} = A + b_{2n,\theta}, \quad W_{2n,\theta} = V_n + \sum \beta_{2n,\theta,jk}B_{jk} + R_{2n,\theta}. \quad (2.45)$$

Here $W_{2n,\theta}$ satisfies, with a constant C independent of n, that

$$W_{2n,\theta}(x) \ge -Cn^2, \quad n = 1,2,\dots, \quad x\in\mathbb{R}^d. \quad (2.46)$$

Indeed, for $|x|\le 2n$, we have $\varphi_{2n}(x)=1$ and (2.16), (2.35) and (2.36) imply

$$\begin{aligned} W_{2n,\theta} &= V + \theta\sum\beta_{jk}B_{jk}(x) + \theta^2 R \\ &\ge V + \theta(|B|-1) + \theta^2 R \ge -C\langle x\rangle^2 \ge -Cn^2; \end{aligned}$$

for $2n < |x| \le 4n$, we have $V_n(x) = 0$ and

$$W_{2n,\theta} = \theta\varphi_{2n}(x)\sum\beta_{jk}B_{jk}(x) + R_{2n,\theta}(x) \ge R_{2n,\theta}(x) \ge -Cn^2;$$

and, for $|x|\ge 4n$, $W_{2n,\theta}(x)=0$. It follows by virtue of (2.40) and (2.44) that

$$\begin{aligned} (1-\theta)\|\nabla_{F_{2n,\theta}}u_n\|^2 &\le (G_{2n,\theta}\nabla_{F_{2n,\theta}}u_n, \nabla_{F_{2n,\theta}}u_n) \\ &= ((L_n - W_{2n,\theta})u_n, u_n) \le (L_n u_n, u_n) + Cn^2\|u_n\|^2 \le Cn^2. \quad (2.47) \end{aligned}$$

Since $|b_{2n,\theta}(x)| \le Cn$ by (2.36), we then have

$$\|\nabla_A u_n\| \le \|\nabla_{F_{2n,\theta}}u_n\| + \|b_{2n,\theta}u_n\| \le \|\nabla_{F_{2n,\theta}}u_n\| + Cn\|u_n\| \le Cn \quad (2.48)$$

and $\|(\nabla\varphi_n)\nabla_A u_n\| \le n^{-1}\|\nabla\varphi\|_\infty\|\nabla_A u_n\| \le C$. It follows, since $\nabla\varphi_n = 0$ for $|x|\le n$, that

$$|(f,(\nabla\varphi_n)\nabla_A u_n)| \le C\|f\|_{L^2(|x|\ge n)} \to 0$$

as $n\to\infty$. Thus, the right of (2.43) vanishes and $f=0$ and L is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

If $V_2 \ne 0$, we repeat the argument above, setting $V_n = \chi_{|x|\le 2n}V_1 + V_2$. Since V_2 is of Stummel class, L_n with this V_n is essentially selfadjoint on

$C_0^\infty(\mathbb{R}^d)$ by virtue of Theorem 2.1.2 and it suffices to show $(f, (\nabla\varphi_n)\nabla_A u_n) \to 0$ as $n \to \infty$ for $u_n \in C_0^\infty(\mathbb{R}^d)$ of (2.41). We use identity (2.44) and obtain

$$(1-\theta)\|\nabla_{F_{2n,\theta}} u_n\|^2 \le (L_n u_n, u_n) - (V_2 u_n, u_n) + Cn^2\|u_n\|^2.$$

as in (2.47). This with (2.42) implies as in (2.48) that

$$\|\nabla_A u_n\|^2 \le C(n^2 + |(V_2 u_n, u_n)|).$$

Since V_2 is $-\Delta$-form bounded with bound 0, we have, for any $\varepsilon > 0$,

$$|(|V_2|u, u)| \le \varepsilon\|\nabla|u|\|^2 + C_\varepsilon\|u\|^2 \le \varepsilon\|\nabla_A u\|^2 + C_\varepsilon\|u\|^2, \quad u \in C_0^\infty(\mathbb{R}^d).$$

It follows that $\|\nabla_A u_n\| \le Cn$ and $\lim_{n\to\infty}(f, (\nabla\varphi_n)\nabla_A u_n) = 0$ as previously. Thus, L is essentially selfadjoint when $V_2 \neq 0$ as well. The closure of L is given by $H = L^*$ and it is standard that $D(L^*) = \{u \in \mathcal{H}\colon -\nabla_A^2 u + Vu \in L^2\}$ and this completes the proof. □

Proof of Theorem 2.1.8 We let θ and $\tilde{V}_1$ be as in the theorem. Define

$$G_{\theta_0} = \mathbf{1} + i\theta_0\beta, \quad F_{\theta_0} = A + \theta_0 b \quad \text{for } \theta \le \theta_0 \le 1$$

by replacing β and b by $\theta_0\beta$ and $\theta_0 b$ in (2.31) and (2.29) respectively. We have

$$-\nabla_A^2 + \tilde{V}_1 = -\nabla_{\overline{F_{\theta_0}}} G_{\theta_0} \nabla_{F_{\theta_0}} + \tilde{W}_{\theta_0}, \quad \tilde{W}_{\theta_0} = \tilde{V}_1 + \theta_0 \sum \beta_{jk} B_{jk} + {\theta_0}^2 R. \quad (2.49)$$

We take the constant $C_1 \ge 10$ large enough in the definition (2.17) of $\tilde{V}_1$ so that $|R(x)| \le 10^{-2} C_1 \langle x\rangle^2$ and

$$\tilde{W}_\theta \ge \tilde{V}_1 + \theta(|B| - 1) + \theta^2 R \ge C_1\langle x\rangle^2 - 1 - |R| \ge \tfrac{2}{3} C_1 \langle x\rangle^2 + 2|R|. \quad (2.50)$$

We show that, for $\theta < \theta_0 \le 1$, there exist a θ_0-dependent constant $C_{\theta_0} > 0$ and a θ_0-independent $C > 0$ such that

$$C_{\theta_0}(|B(x)| + |\tilde{V}_1(x)|) + \tfrac{C_1}{2}\langle x\rangle^2 \le \tilde{W}_{\theta_0}(x) \le (|B(x)| + |\tilde{V}_1(x)| + C\langle x\rangle^2). \quad (2.51)$$

Indeed, the second inequality is obvious from (2.36). The first is also evident if $\tilde{V}_1 > 0$, since then $\tilde{V}_1 + \theta|B| \ge C_1\langle x\rangle^2$ and

$$\tilde{W}_{\theta_0} \ge \tilde{V}_1 + \theta_0(|B| - 1) + \theta_0^2 R \ge \frac{1}{2}(|\tilde{V}_1| + \theta_0|B| + C_1\langle x\rangle^2).$$

To see the first for the case $\tilde{V}_1(x) < 0$, we first estimate

$$\tilde{W}_{\theta_0} = \tilde{W}_\theta + (\theta_0 - \theta)\sum \beta_{jk} B_{jk} + (\theta_0^2 - \theta^2) R$$

$$\geq \tfrac{2}{3}C_1\langle x\rangle^2 + (\theta_0-\theta)(|B|-1) - |R| \geq \tfrac{1}{2}C_1\langle x\rangle^2 + (\theta_0-\theta)|B|$$

which holds irrespectively of the sign of $\tilde{V}_1$. If $\tilde{V}_1(x) < 0$ we also have

$$\begin{aligned}\tilde{W}_{\theta_0} &= \tfrac{\theta_0}{\theta}\tilde{W}_\theta + \left(\tfrac{\theta_0}{\theta}-1\right)|\tilde{V}_1| + \theta_0(\theta_0-\theta)R \\ &\geq \tfrac{\theta_0}{\theta}(\tfrac{2}{3}C_1\langle x\rangle^2 + 2|R|) + \left(\tfrac{\theta_0}{\theta}-1\right)|\tilde{V}_1| - |R| \geq \tfrac{2}{3}C_1\langle x\rangle^2 + \left(\tfrac{\theta_0}{\theta}-1\right)|\tilde{V}_1|.\end{aligned}$$

Adding both sides of last two estimates and dividing by 2, we obtain the first inequality of (2.51) for the case $\tilde{V}_1(x) < 0$.

We define the quadratic form $q_1(u,v)$ for $u, v \in C_0^\infty(\mathbb{R}^d)$ by

$$q_1(u,v) = (\nabla_A u, \nabla_A v) + (\tilde{V}_1 u, v). \tag{2.52}$$

We have by virtue of Iwatsuka's identity (2.49) for θ_0 replacing θ that

$$q_1(u,v) = (G_{\theta_0}\nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v) + (\tilde{W}_{\theta_0} u, v). \tag{2.53}$$

Estimates $1-\theta_0 \leq G_{\theta_0} \leq 1+\theta_0$ and (2.51) imply for a constant $C > 1$ that

$$\begin{gathered}(1-\theta_0)\|\nabla_{F_{\theta_0}} u\|^2 \leq (G_{\theta_0}\nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} u) \leq (1+\theta_0)\|\nabla_{F_{\theta_0}} u\|^2, \\ C^{-1}\|(|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u\|^2 \leq (\tilde{W}_{\theta_0} u, u) \leq C\|(|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u\|^2.\end{gathered}$$

It follows that quadratic forms $(G_{\theta_0}\nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v)$ and $(\tilde{W}_{\theta_0} u, v)$ on $C_0^\infty(\mathbb{R}^d)$ are both closable and positive definite and their closures have respective domains $\{u\colon \nabla_{F_{\theta_0}} u \in L^2\}$ and $\{u\colon (|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u \in L^2\}$. Thus, q_1 is closable, the closure $[q_1]$ has domain

$$\begin{aligned} D([q_1]) &= \{u \in L^2\colon \nabla_{F_{\theta_0}} u \in L^2,\ (|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u \in L^2\} && (2.54)\\ &= \{u \in L^2\colon \nabla_A u \in L^2,\ (|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u \in L^2\} && (2.55)\end{aligned}$$

and $[q_1](u)$ is given again by (2.52). Moreover, by making C_1 larger if necessary, we have from the first inequality of (2.51) and that $|b_{\theta_0}| \leq C\langle x\rangle$ that

$$[q_1](u) \geq (1-\theta_0)\|\nabla_A u\|^2 + C\|(|B| + |\tilde{V}_1| + \langle x\rangle^2)^{\frac{1}{2}} u\|^2, \quad u \in D([q_1]). \tag{2.56}$$

We have $q_0(u,v) = q_1(u,v) + (V_2 u, v)$. Since V_2 is of Kato-class, V_2 is $-\Delta$-form bounded with bound 0 and we have, for any $\varepsilon > 0$,

$$(|V_2|u,u) \leq \varepsilon\|\nabla_A u\|^2 + C_\varepsilon\|u\|^2 \tag{2.57}$$

as in the proof of Theorem 2.1.7. Hence the form $(|V_2|u,u)$ is $[q_1]$-bounded with bound 0 and statements (1) and (2) of the theorem follow.

We prove statement (3). We write $\tilde{V} = \tilde{V}_1 + V_2$. Let $u \in D(H_0)$. Then, $u \in D([q_0])$ and $\langle x \rangle u$, $|\tilde{V}_1|^{\frac{1}{2}} u, |V_2|^{\frac{1}{2}} u \in \mathcal{H}$ and $\nabla_A u \in \mathcal{H}$. Hence, $\tilde{V} u \in L^1_{\text{loc}}$ and $\nabla_A^2 u$ is well defined as distributions. It follows for any $v \in C_0^\infty(\mathbb{R}^d)$ that

$$(H_0 u, v) = [q_0](u, v) = (\nabla_A u, \nabla_A v) + (\tilde{V} u, v) = (-\nabla_A^2 u + \tilde{V} u, v).$$

Hence $-\nabla_A^2 u + \tilde{V} u \in L^2$ and $H_0 u = -\nabla_A^2 u + \tilde{V} u$. Suppose on the contrary that $u \in D([q_0])$ satisfies $-\nabla_A^2 u + \tilde{V} u \in L^2$. Then, for any $v \in C_0^\infty(\mathbb{R}^d)$,

$$(-\nabla_A^2 u + \tilde{V} u, v) = [q_0](u, v) = (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v) + ((\tilde{W}_{\theta_0} + V_2) u, v)$$

and this extends to all $v \in D([q_0])$ by virtue of the argument in the first part. Thus, $u \in D(H_0)$ and $H_0 u = -\nabla_A^2 u + \tilde{V} u$. This completes the proof. □

The following is a corollary of the proof of Theorem 2.1.8.

Corollary 2.3.2. *Let conditions of* Theorem 2.1.8 *be satisfied. Let C_1 be sufficiently large. Then, for a constant $C > 0$, we have*

$$\|\nabla_A u\|^2 + \|(|B| + |\tilde{V}_1| + \langle x \rangle^2)^{\frac{1}{2}} u\|^2 \leq C[q_0](u), \quad u \in D([q_0]) \tag{2.58}$$

2.4 Diamagnetic inequality

In this section we assume that A and V satisfy the following conditions:

(1) $A(x) \in C^3(\mathbb{R}^d)$ and $B(x)$ satisfies estimates (2.15).

(2) $V = V_1 + V_2$ with $V_1 \in L^1_{\text{loc}}$ and V_2 of Kato class.

(3) There exists constants $0 < \theta < 1$, $C_* > 1$ and $Q \in M(\mathbb{R}^d)$ such that

$$\theta |B(x)| + V_1(x) + C_* \langle x \rangle^2 \geq Q(x)^2. \tag{2.59}$$

We then define $q_0(u)$ and $q_1(u)$ respectively by (2.18) and (2.52) with $\tilde{V}_1(x) = V_1(x) + (C_* + C_1)\langle x \rangle^2$ with sufficiently large constant C_1 such that results in the previous section are satisfied. We let H_0 and H_1 be selfadjoint operators defined by $[q_0]$ and $[q_1]$ respectively.

Lemma 2.4.1. *Let $\theta < \theta_0 < 1$. There exists $C_{\theta_0} > 0$ such that for $C_1 \geq C_{\theta_0}$, we have the following estimate:*

$$(1 - \theta_0)\|\nabla_{F_{\theta_0}} u\|^2 + \|Q^2 u\|^2 + 2(\theta_0 - \theta)\|Q|B|^{\frac{1}{2}} u\|^2 + C_1 \|\langle x \rangle Q u\|^2 \leq \|H_1 u\|^2, \quad u \in D(H_1). \tag{2.60}$$

Proof. We use the notation of the proof of Theorem 2.1.8. We have as in there

$$\tilde{W}_{\theta_0} \geq Q(x)^2 + (\theta_0 - \theta)|B(x)| + \tfrac{2}{3}C_1\langle x\rangle^2 \tag{2.61}$$

Let $u \in D(H_1)$. Then, $\nabla_{F_{\theta_0}}u$, $\nabla_A u$, $\tilde{W}_{\theta_0}^{1/2}u$ and Qu all belong to $L^2(\mathbb{R}^d)$ by virtue of (2.61) and, for $v \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}(G_{\theta_0}Q\nabla_{F_{\theta_0}}u, Q\nabla_{F_{\theta_0}}v) &= -(\nabla_{\overline{F_{\theta_0}}}G_{\theta_0}\nabla_{F_{\theta_0}}u, Q^2 v) - (G_{\theta_0}\nabla_{F_{\theta_0}}u, \nabla(Q^2)v)\\ &= (H_1 u, Q^2 v) - (\tilde{W}_{\theta_0}u, Q^2 v) - (G_{\theta_0}\nabla_{F_{\theta_0}}u, \nabla(Q^2)v).\end{aligned} \tag{2.62}$$

Using $\varphi_n(x)$ of the proof of Theorem 2.1.7 and Friedrich's mollifier j_ε, we define $v_{\varepsilon,n} = j_\varepsilon * (\varphi_n^2 u)$ for $0 < \varepsilon < 1$ and $n = 1, 2, \ldots$. Then, $v_{\varepsilon,n} \in C_0^\infty(\mathbb{R}^d)$, is supported by the ball $B_{2(n+1)}(0)$ and $v_{\varepsilon,n} \to \varphi_n^2 u$ in the Sobolev space $H^1(\mathbb{R}^d)$ as $\varepsilon \to 0$. We replace v in (2.62) by $v_{\varepsilon,n}$, rewrite the left hand side of the resulting equation as $(G_{\theta_0}\varphi_n Q\nabla_{F_{\theta_0}}u, \varphi_n Q\nabla_{F_{\theta_0}}u) + 2(\varphi_n G_{\theta_0}Q\nabla_{F_{\theta_0}}u, Q(\nabla\varphi_n)u)$ and arrange it as follows:

$$\begin{aligned}(G_{\theta_0}\varphi_n Q\nabla_{F_{\theta_0}}u, \varphi_n Q\nabla_{F_{\theta_0}}u) + (\tilde{W}_{\theta_0}u, Q^2\varphi_n^2 u) = (H_1 u, Q^2\varphi_n^2 u)\\ - 2(\varphi_n G_{\theta_0}Q\nabla_{F_{\theta_0}}u, Q(\nabla\varphi_n)u) - (G_{\theta_0}Q\nabla_{F_{\theta_0}}u, Q^{-1}\nabla(Q^2)\varphi_n^2 u)\end{aligned} \tag{2.63}$$

By virtue of (2.61) the left hand side may be bounded from below by

$$(1-\theta_0)\|\varphi_n Q\nabla_{F_{\theta_0}}u\|^2 + \|\varphi_n Q^2 u\|^2 + (\theta_0 - \theta)\|\varphi_n Q|B|^{\frac{1}{2}}u\|^2 + \tfrac{2C_1}{3}\|\varphi_n\langle x\rangle Qu\|^2. \tag{2.64}$$

The right hand side of (2.63) may be bounded from above by

$$\begin{aligned}\|\varphi_n H_1 u\|\|\varphi_n Q^2 u\| + 4n^{-1}\|\nabla\varphi\|_\infty\|\varphi_n Q\nabla_{F_{\theta_0}}u\|\|Qu\|\\ + 4\|\varphi_n Q\nabla_{F_{\theta_0}}u\|\|\varphi_n(\nabla Q)u\|.\end{aligned} \tag{2.65}$$

Here we have $\|\varphi_n(\nabla Q)u\| \leq C_Q\|\varphi_n\langle x\rangle Qu\|$ since $Q \in M(\mathbb{R}^d)$, and we further estimate (2.65) from above by

$$\begin{aligned}\tfrac{1}{2}\|\varphi_n H_1 u\|^2 + \tfrac{1}{2}\|\varphi_n Q^2 u\|^2 + 2n^{-1}\|\nabla\varphi\|_\infty(\|\varphi_n Q\nabla_{F_{\theta_0}}u\|^2 + \|Qu\|^2)\\ + \tfrac{1-\theta_0}{2}\|\varphi_n Q\nabla_{F_{\theta_0}}u\|^2 + \tfrac{8C_Q^2}{1-\theta_0}\|\varphi_n\langle x\rangle Qu\|^2.\end{aligned} \tag{2.66}$$

Combining (2.64) and (2.66), we conclude that

$$\begin{aligned}\left(\tfrac{1-\theta_0}{2} - \tfrac{2\|\nabla\varphi\|_\infty}{n}\right)\|\varphi_n Q\nabla_{F_{\theta_0}}u\|^2 + \tfrac{1}{2}\|\varphi_n Q^2 u\|^2 + (\theta_0 - \theta)\|\varphi_n Q|B|^{\frac{1}{2}}u\|^2\\ + \left(\tfrac{2C_1}{3} - \tfrac{8C_Q^2}{1-\theta_0}\right)\|\varphi_n\langle x\rangle Qu\|^2 \leq \tfrac{1}{2}\|H_1 u\|^2 + \tfrac{2}{n}\|\nabla\varphi\|_\infty\|Qu\|^2.\end{aligned}$$

We choose $C_1 > 0$ larger if necessary so that

$$\tfrac{C_1}{6} \geq \tfrac{8C_Q^2}{1-\theta_0}$$

and let $n \to \infty$. Then the monotone convergence implies that Q^2u, $Q\nabla_{F_{\theta_0}}u$, $Q|B|^{\frac{1}{2}}u$ and, a fortiori $\langle x\rangle Qu$ all belong to $L^2(\mathbb{R}^d)$ and we obtain (2.60). □

Since $F_{\theta_0} = A + \theta_0 b$ and $|b| \leq C\langle x\rangle$, we have

$$(1-\theta_0)\|Q\nabla_A u\|^2 \leq 2(1-\theta_0)\|Q\nabla_{F_{\theta_0}}u\|^2 + 2C^2(1-\theta_0)\theta_0^2\|\langle x\rangle Qu\|^2.$$

Thus, assuming $2C^2 < C_1$, we obtain the following Corollary.

Corollary 2.4.2. *For $\theta < \theta_0 < 1$, there exists $C_{\theta_0} > 0$ such that for $C_1 \geq C_{\theta_0}$*

$$\begin{aligned}(1-\theta_0)\|Q\nabla_A u\|^2 + \|Q^2u\|^2 \\ + 2(\theta_0-\theta)\|Q|B|^{\frac{1}{2}}u\|^2 + C_1\|\langle x\rangle Qu\|^2 \leq 2\|H_1u\|^2, \quad u \in D(H_1).\end{aligned} \tag{2.67}$$

Write $a_\pm = \max(0, \pm a)$ and define non-negative quadratic form:

$$q_{1+}(u) = \|\nabla_A u\|^2 + \|\tilde{V}_{1+}^{\frac{1}{2}}u\|^2, \quad D(q_{1+}) = C_0^\infty(\mathbb{R}^d).$$

Theorem 2.1.8 implies that q_{1+} is closable and we denote by $H_{1+} = -\nabla_A^2 + \tilde{V}_{1+}$ the selfadjoint operator defined by $[q_{1+}]$.

Lemma 2.4.3. *For any $\theta < \theta_0 < 1$, there exists C_{θ_0} such that, for $C_1 > C_{\theta_0}$ we have*

$$\|\tilde{V}_{1-}u\| \leq (\theta/\theta_0)\|H_{1+}u\|, \quad u \in D(H_{1+}). \tag{2.68}$$

It follows, particular, that $D(H_1) = D(H_{1+})$.

Proof. Let $\theta < \theta_0 < 1$. Since $\tilde{V}_{1+}(x) \geq 0$, we obviously have

$$\theta_0|B(x)| + \tilde{V}_{1+}(x) + C_*\langle x\rangle^2 \geq \theta_0(1+|B|^2+x^4)^{1/2}$$

and assumption (2.15) implies $Q_0(x) = \theta_0^{\frac{1}{2}}(1+|B|^2+x^4)^{1/4} \in M(\mathbb{R}^d)$. Then, take θ_1 such that $\theta_0 < \theta_1 < 1$ and repeat the argument of the proof of Lemma 2.4.1 using H_{1+}, θ_0, θ_1 and Q_0 in place of H_1, θ, θ_0 and Q respectively. We obtain from (2.60) that, for $C_1 > C_{\theta_0}$,

$$\|Q_0^2(x)u\| \leq \|H_{1+}u\|, \quad u \in D(H_{1+}). \tag{2.69}$$

Since $\tilde{V}_{1-} \leq \theta|B(x)|$ by virtue of (2.59) and $\theta|B(x)| \leq (\theta/\theta_0)Q_0^2(x)$, (2.69) implies the lemma. □

Theorem 2.4.4. *There exist uniformly bounded operators $B_a \in \mathbf{B}(\mathcal{H})$ for $a > 0$ such that, for every $u \in L^2(\mathbb{R}^d)$, we have*

$$|(H_1 + a^2)^{-1}u(x)| \leq (H_{1+} + a^2)^{-1}|B_a u|(x) \leq (-\Delta + a^2)^{-1}|B_a u|(x). \quad (2.70)$$

Proof. Lemma 2.4.3 implies that, for any $\theta < \theta_0 < 1$, provided that $C_1 \geq C_{\theta_0}$,

$$\|\tilde{V}_{1-}(H_{1+} + a^2)^{-1}u\| \leq (\theta/\theta_0)\|u\|, \quad u \in L^2$$

for any $a > 0$. It follows that

$$(H_1 + a^2)^{-1} = (H_{1+} + a^2)^{-1}B_a, \quad B_a = (\mathbf{1} - \tilde{V}_{1-}(H_{1+} + a^2)^{-1})^{-1} \quad (2.71)$$

and $\|B_a\| \leq (1 - (\theta/\theta_0))^{-1}$. We then apply the diamagnetic inequality (pp. 9–10 of [5]) to $H_{1+} + a^2$. The lemma follows. □

Corollary 2.4.5. *Provided that C_1 is large enough, we have*

$$\|(-\Delta + 1)^{1/2}Q|u|\| \leq C\|H_1 u\|, \quad u \in D(H_1). \quad (2.72)$$

Proof. Corollary 2.4.2 implies $Qu \in L^2$ and $\nabla_A(Qu) = Q\nabla_A u + (\nabla Q)u \in L^2$. It follows, since $|\nabla|u|| \leq |\nabla_A u|$, that $Q|u| \in H^1$ and

$$\|(-\Delta+1)^{1/2}Q|u|\|^2 = \|Qu\|^2 + \|\nabla|Qu|\|^2 \leq \|Qu\|^2 + \|\nabla_A(Qu)\|^2 \leq C\|H_1 u\|^2.$$

Estimate (2.72) follows. □

2.5 Proof of Theorem 2.1.9

.

In this and next sections we prove Theorems 2.1.9 and 2.1.10 respectively. Before starting the proof, we briefly discuss the gauge transform which will play an important role in what follows. We define the gauge transform by

$$v(t,x) = G(t)u(t,x) = e^{-iF(t)\langle x\rangle^2}u(t,x), \quad F(t) = \int_0^t (C(s) + C_1)ds \quad (2.73)$$

by using a strongly continuous family of unitary operators $G(t)$, where $C_1 > 0$ a large constant. Then, $u(t,x)$ satisfies (2.1) if and only if $v(t,x)$ does

$$i\partial_t v = (-\nabla^2_{\tilde{A}(t)}v + \tilde{V}(t,x))v, \quad (2.74)$$

$$\tilde{A}(t,x) = A(t,x) - 2F(t)x, \quad \tilde{V}(t,x) = V(t,x) + (C(t) + C_1)\langle x\rangle^2 \quad (2.75)$$

and, provided a dense subspace Σ satisfies $G(t)\Sigma = \Sigma$, $\{U(t,s) \colon t,s \in \mathbb{R}\}$ is a unitary propagator for (2.1) on Σ if and only if so is

$$\tilde{U}(t,s) = G(t)U(t,s)G(s)^{-1} \tag{2.76}$$

for (2.74) on Σ. If V_1 satisfies (2.16), $\tilde{V}_1(t,x) = V_1(t,x) + (C(t)+C_1)\langle x\rangle^2$ does

$$|B(t,x)| + \tilde{V}_1(t,x) \geq Q(x)^2 + C_1\langle x\rangle^2. \tag{2.77}$$

We assume in what follows that $C_1 > 0$ is taken sufficiently large so that, with this $\tilde{V}_1(t,x)$, Theorems 2.1.7 and 2.1.8 as well as Lemma 2.4.1 and Theorem 2.4.4 are satisfied uniformly with respect to $t \in I$. In the proof, we shall first construct propagator $\tilde{U}(t,s)$ for equation (2.74), define $U(t,s)$ by (2.76) and check that it satisfies the properties of Theorem 2.1.9 or Theorem 2.1.10.

We now begin the proof of Theorem 2.1.9. We consider five operators

$$L(t) = -\nabla^2_{A(t)} + V(t),\ L_0(t) = -\nabla^2_{A(t)} + \tilde{V}(t),\ L_1(t) = -\nabla^2_{A(t)} + \tilde{V}_1(t),$$
$$\tilde{L}_0(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}(t), \quad \tilde{L}_1(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}_1(t).$$

These operators are all essentially selfajoint on $C_0^\infty(\mathbb{R}^d)$ and we denote their selfadjoit extensions by $H(t)$, $H_0(t)$, $H_1(t)$, $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$, respectively.

Since $V_2(t,x)$ is of Stummel class uniformly with respect to $t \in I$, Theorem 2.4.4 implies that, for any $\varepsilon > 0$, there exists a_0 such that

$$\|V_2(t)(H_1(t)+a^2)^{-1}\|_{\mathbf{B}(\mathcal{H})} \leq \|V_2(t)(-\Delta+a^2)^{-1}\|_{\mathbf{B}(\mathcal{H})}\|B_a\|_{\mathbf{B}(\mathcal{H})} < \varepsilon, \quad a > a_0.$$

It follows by Kato-Rellich theorem that

$$H_0(t) = H_1(t) + V_2(t), \quad D(H_0(t)) = D(H_1(t)). \tag{2.78}$$

Moreover, by choosing C_1 large enough we may assume by virtue of (2.60),

$$\|u\| \leq \|H_1(t)u\|, \quad \|V_2(t)H_1(t)^{-1}\| \leq 1/2, \quad t \in I.$$

Then, we have for a constant C_0

$$C_0^{-1}\|H_1(t)u\| \leq \|H_0(t)u\| \leq C_0\|H_1(t)u\|, \quad t \in I. \tag{2.79}$$

Since $\tilde{A}$ and A produce the same magnetic field and $|\tilde{A} - A| \leq C\langle x\rangle$, (2.78) holds with $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$ in place of $H_0(t)$ and $H_1(t)$ respectively and we likewise have

$$C_0^{-1}\|\tilde{H}_1(t)u\| \leq \|\tilde{H}_0(t)u\| \leq C_0\|\tilde{H}_1(t)u\|. \tag{2.80}$$

Lemma 2.5.1. (1) *Domains of* $H_0(t)$, $H_1(t)$, $\tilde{H}_0(t)$ *and* $\tilde{H}_1(t)$ *satisfy*

$$D(H_0(t)) = D(H_1(t)) = D(\tilde{H}_0(t)) = D(\tilde{H}_1(t)) \equiv \mathcal{D} \subset D(H(t))$$

for all $t \in I$ *and* $\mathcal{D}$ *is independent of* $t \in I$.

(2) *There exists a constant* $c > 0$ *such that*

$$\|H_0(t)u\| \le e^{c|t-s|}\|H_0(s)u\|, \quad t, s \in I, \tag{2.81}$$
$$\|(H_0(t) - H_0(s))u\| \le c|t-s|\|H_0(s)u\|, \quad t, s \in I. \tag{2.82}$$

The same holds for $\tilde{H}_0(t)$ *replacing* $H_0(t)$.

(3) *The gauge transform* $G(t) = e^{-iF(t)\langle x\rangle^2}$ *satisfies* $G(t)\mathcal{D} = \mathcal{D}$ *and*

$$G(t)H_0(t) = \tilde{H}_0(t)G(t), \quad G(t)H_1(t) = \tilde{H}_1(t)G(t). \tag{2.83}$$

If $\varphi \in \mathcal{D}$, $t \mapsto G(t)\varphi$ *is* $\mathcal{D}$*-valued continuous,* $\mathcal{H}$*-valued* C^1 *and*

$$\partial_t G(t)\varphi = -i(C(t) + C_1)\langle x\rangle^2 G(t)\varphi.$$

Proof. We write $C(t)$ for $C(t)+C_1$ in the proof by absorbing C_1 into $C(t)$ for shorting formulas. Let $u \in C_0^\infty(\mathbb{R}^d)$. Then, $H_0(t)u$ is $\mathcal{H}$-valued differentiable almost everywhere with respect to t and

$$\dot{H}_0(t)u = 2i\dot{A}(t,x)\nabla_{A(t)}u + i\nabla_x \cdot \dot{A}(t,x)u + \dot{C}(t)\langle x\rangle^2 + \dot{V}(t,x)u. \tag{2.84}$$

We write the right hand side in the form

$$2i\dot{A}(t,x)\cdot\nabla_{A(s,x)}u + 2\dot{A}(t,x)\cdot\left(\int_s^t \dot{A}(r,x)dr\right)u + (i\nabla_x\cdot\dot{A}(t,x) + \dot{C}(t)\langle x\rangle^2)u$$
$$+ \dot{V}(t,x)u = I_1(t,s)u + I_2(t,s)u + I_3(t)u + I_4(t)u.$$

Since $|\dot{A}(t,x)| \le CQ(x)$, (2.67) implies

$$\|I_1(t,s)u\| \le 2\||\dot{A}(t,x)||\nabla_{A(s)}u|\| \le C\|Q\nabla_{A(s)}u\| \le C\|H_1(s)u\|.$$

Denote by $M(t,x)$ any of $\nabla_x(\dot{A}(t,x)^2)$, $\dot{A}(t,x)^2$, $\nabla_x \cdot \dot{A}(t,x)$ and $\dot{C}(t)\langle x\rangle^2$. Then, $|M(t,x)| \le CQ(x)^2$ and (2.67) implies $\|M(t)H_1(s)^{-1}u\| \le C_1\|u\|$ uniformly with respect to $t, s \in I$. Thus,

$$\|I_2(t,s)u\| + \|I_3(t)u\| \le C\|H_1(s)u\|, \quad t, s \in I.$$

Write $\dot{V}(t,x) = W_0(t,x) + W_1(t,x) + W_2(t,x)$ as in Theorem 2.1.9, then $\|W_0(t)u\| \le C\|Q^2 u\| \le C\|H_1(s)u\|$ for any $t, s \in I$ as above;

$$\|W_1(t)u\| \le \|Q^{-1}W_1(t)(-\Delta+1)^{-\frac{1}{2}}\|_{\mathbf{B}(\mathcal{H})}\|(-\Delta+1)^{\frac{1}{2}}Q|u|\| \le C\|H_1(s)u\|$$

by virtue of (2.72); and Theorem 2.4.4 implies

$$\|W_2(t)H_1(s)^{-1}u\| \le C\|W_2(t)(-\Delta+1)^{-1}|B_1 u|\| \le C\|B_1 u\| \le C\|u\|.$$

Thus, $\|I_4(t,s)u\| \le C\|H_1(s)u\|$ and combining these estimates, we obtain

$$\|\dot{H}_0(t)u\| \le C\|H_1(s)u\| \le C\|H_0(s)u\|, \quad t, s \in I. \tag{2.85}$$

It follows by integration that

$$\|(H_0(t) - H_0(s))u\| \le c|t-s|\|H_0(s)u\|, \quad u \in C_0^\infty(\mathbb{R}^d). \tag{2.86}$$

Since $C_0^\infty(\mathbb{R}^d)$ is a core of $H_0(s)$, (2.86) extends to $u \in D(H_0(s))$. It follows that $D(H_0(s)) \subset D(H_0(t))$ and by symmetry $D(H_0(s)) = D(H_0(t))$ for any $t, s \in I$ and, consequently, (2.82) for $H_0(t)$ is satisfied. (2.82) clearly implies (2.81). Changing $A(t)$ by $\tilde{A}(t)$ will not change $B(t,x)$ and the argument above yields the same results for $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$. This proves statement (2).

Let $u \in D(H_0(t))$. Then, $\langle x\rangle^2 u \in \mathcal{H}$ by virtue of (2.67) and

$$H(t)u = H_0(t)u - C(t)\langle x\rangle^2 u \in \mathcal{H}. \tag{2.87}$$

Since $D(H(t)) = \{u \in \mathcal{H} \colon H(t)u \in L^2\}$, (2.87) implies $u \in D(H(t))$ and $D(H_0(t)) \subset D(H(t))$.

We next prove $D(H_1(t)) = D(\tilde{H}_1(t))$, which will then prove statement (1). Define for $\theta \in [0,1]$

$$H_1(t,\theta) = -\nabla^2_{A(t,\theta)} + \tilde{V}_1(t,x), \quad A(t,\theta,x) = A(t,x) - 2\theta F(t)x,$$

so that $H_1(t,0) = H_1(t)$ and $H_1(t,1) = \tilde{H}_1(t)$. Since $A(t,\theta,x)$ and $A(t,x)$ generate the same magnetic field $B(t,x)$ and $|2\theta F(t)x| \le C\langle x\rangle$, results of previous sections apply to $H_1(t,\theta)$. We have

$$\partial_\theta H_1(t,\theta)u = -i4F(t)x\nabla_{A(t)}u + 8\theta F(t)^2 x^2 u - 2diF(t)u$$

and (2.67) implies $\|\partial_\theta H_1(t,\theta)u\| \le C\|H_1(t)u\|$ for $0 \le \theta \le 1$. Thus,

$$\|(H_1(t,\theta) - H_1(t,\sigma))u\| \le C|\theta-\sigma|\|H_1(t,\sigma)u\|, \quad u \in C_0^\infty(\mathbb{R}^d),$$

and we obtain the desired result $D(H_1(t)) = D(\tilde{H}_1(t))$ as previously.

It is clear that $G(t)$ is an isomorphism of $C_0^\infty(\mathbb{R}^d)$ and $G(t)H_0(t)\varphi = \tilde{H}_0(t)G(t)\varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is a core of $H_0(t)$, it follows that $G(t)D(H_0(t)) \subset D(\tilde{H}_0(t))$. This clearly holds for $G(-t) = G(t)^{-1}$ as well and we obtain $G(t)\mathcal{D} = \mathcal{D}$ and $G(t)H_0(t) = \tilde{H}_0(t)G(t)$. This argument likewise applies to the pair $H_1(t)$ and $\tilde{H}_1(t)$ and we obtain (2.83). The last statement is obvious since $\mathcal{D} \subset D(\langle x\rangle^2)$. This completes the proof. □

Proof of Theorem 2.1.9. Lemma 2.5.1 yields statement (a) of the theorem. It also implies that graph norms of any two of $\{H_0(t), \tilde{H}_0(s) \colon t, s \in I\}$ are equivalent to each other. We equip $\mathcal{D}$ with the graph norm of $H_0(t_0)$ as in the theorem. Then, it is obvious that $\mathcal{D} \subset \mathcal{H}$ continuously and densely, $\mathcal{D} = D(\tilde{H}_0(t))$ for every $t \in I$ and that $I \ni t \mapsto \tilde{H}_0(t) \in \mathbf{B}(\mathcal{D}, \mathcal{H})$ is norm continuous by virtue of (2.82) for $\tilde{H}_0(t)$. We wish to apply Theorem 2.2.1 to the triplet $(\mathcal{X}, \mathcal{Y}, A(t))$ by setting $\mathcal{X} = \mathcal{H}$, $\mathcal{Y} = \mathcal{D}$ and $A(t) = \tilde{H}_0(t)$. For this we need check conditions (1) and (2) of Theorem 2.2.1 are satisfied.

For $t \in I$, we define $\mathcal{Y}_t = \mathcal{D}$ but with the graph norm of $\tilde{H}_0(t)$ and $\mathcal{X}_t = \mathcal{H}$. Then, the norm of $\mathcal{Y}_t$ is equivalent to that of $\mathcal{D}$ and (2.81) for $\tilde{H}_0(t)$ implies condition (2.27). It follows from Theorem 2.1.2 that $\tilde{H}_0(t)$ is selfadjoint in $\mathcal{X}_t = \mathcal{H}$. Hence the part of $\tilde{H}_0(t)$ in $\mathcal{Y}_t (= D(\tilde{H}_0(t)))$ is automatically selfadjoint with domain $D(\tilde{H}_0(t)^2)$. Thus, the conditions are satisfied.

It follows that there uniquely exists a family of operators $\{\tilde{U}(t,s) \colon s, t \in I\}$ which satisfies properties of Theorem 2.2.1 for $(\mathcal{H}, \mathcal{D}, \tilde{H}_0(t))$. Moreover, $\tilde{U}(t,s)$ is a unitary operator of $\mathcal{H}$. Indeed, if we set $u(t) = \tilde{U}(t,s)\varphi$ for $\varphi \in \mathcal{Y}$, $i\partial_t \|u(t)\|^2 = (\tilde{H}_0(t)u(t), u(t)) - (u(t), \tilde{H}_0(t)u(t)) = 0$ since $\tilde{H}_0(t)$ is selfadjoint. Hence $\tilde{U}(t,s)$ is an isometry of $\mathcal{H}$ and, since $\tilde{U}(t,s)\mathcal{D} = \mathcal{D}$, it is unitary. We define

$$U(t,s) = G(t)^{-1}\tilde{U}(t,s)G(s).$$

Then, $U(t,s)$ is a strongly continuous family of unitary operators on $\mathcal{H}$; Lemma 2.5.1 (3) implies that $U(t,s) \in \mathbf{B}(\mathcal{D})$; if $\varphi \in \mathcal{D}$, $U(t,s)\varphi$ is $\mathcal{D}$-valued continuous, $\mathcal{H}$-valued C^1 and that $U(t,s)\varphi$ satisfies the first of Eqns. (2.10):

$$i\partial_t U(t,s)\varphi = G(t)^{-1}(-C(t)\langle x\rangle^2 + \tilde{H}_0(t))\tilde{U}(t,s)G(s)\varphi = H(t)U(t,s)\varphi.$$

We may similarly prove that $U(t,s)\varphi$ satisfies the other of (2.10).

For proving the uniqueness of $U(t,s)$ we have only to notice the following: If $U(t,s)$ satisfies properties of the theorem, then $\tilde{U}(t,s) = G(t)U(t,s)G(s)^{-1}$ does those for $\tilde{H}_0(t)$ and such $\tilde{U}(t,s)$ is unique by virtue of Theorem 2.2.1.

When $\varphi \in \mathcal{D}$, (2.10) shows that $u(t,x) = U(t,s)\varphi(x)$ satisfies (2.1) in the sense of distributions. Then, the standard approximation argument shows that the same holds for $\varphi \in \mathcal{H}$ as well and $U(t,s)$ is unitary propagator on $\mathcal{H}$ for (2.1). We omit the details. The proof is completed. $\square$

2.6 Proof of Theorem 2.1.10

For the constant θ in (2.24) we take and fix θ_0 such that $\theta < \theta_0 < 1$ and take the constant $C_1 > 0$ large enough so that results of Sections 2.3 and 2.4 are satisfied, uniformly with respect to $t \in I$, for $q_0(t)$ of (2.26) and

$$q_1(t)(u,v) = (\nabla_{A(t)}u, \nabla_{A(t)}v) + (\tilde{V}_1(t)u, v), \quad u, v \in C_0^\infty(\mathbb{R}^d),$$

in place of q_0 and q_1 respectively. In addition to $q_0(t)$ and $q_1(t)$, we define

$$\tilde{q}_0(t)(u,v) = (\nabla_{\tilde{A}(t)}u, \nabla_{\tilde{A}(t)}v) + (\tilde{V}u, v), \quad u, v \in C_0^\infty(\mathbb{R}^d), \tag{2.88}$$

$$\tilde{q}_1(t)(u,v) = (\nabla_{\tilde{A}(t)}u, \nabla_{\tilde{A}(t)}v) + (\tilde{V}_1 u, v), \quad u, v \in C_0^\infty(\mathbb{R}^d), \tag{2.89}$$

where $\tilde{A}(t,x) = A(t,x) - 2F(t)x$. Since $\tilde{A}(t,x)$ and $A(t,x)$ generate same magnetic field and they differ only by $2F(t)x$, results of Sections 2.3 and 2.4 likewise apply to $\tilde{q}_0(t)$ and $\tilde{q}_1(t)$ uniformly for $t \in I$. In particular, since V_2 is of Kato class uniformly with respect to $t \in I$, $\tilde{q}_1(t)$ is uniformly positive definite and

$$C^{-1}\tilde{q}_1(t)(u) \le \tilde{q}_0(t)(u) \le C\tilde{q}_1(t)(u), \quad u \in C_0^\infty(\mathbb{R}^d) \tag{2.90}$$

for a t-independent constant $C > 0$. Thus, $D([q_0(t)]) = D([q_1(t)])$ and $D([\tilde{q}_0(t)]) = D([\tilde{q}_1(t)])$. We denote by $H_0(t)$, $H_1(t)$, $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$ selfadjoint operators defined respectively by $[q_0(t)]$, $[q_1(t)]$, $[\tilde{q}_0(t)]$ and $[\tilde{q}_1(t)]$. As in the previous section, we write $C(t)$ for $C(t) + C_1$ absorbing C_1 into $C(t)$.

Lemma 2.6.1. (1) *Domains of* $[q_0(t)]$, $[q_1(t)]$, $[\tilde{q}_0(t)]$ *and* $[\tilde{q}_1(t)]$ *satisfy*

$$D([q_0(t)]) = D([q_1(t)]) = D([\tilde{q}_0(t)]) = [\tilde{q}_1(t)] = \mathcal{Y} \subset D(L_Q^{\frac{1}{2}})$$

and are independent of $t \in I$.

(2) *There exists a constant* $c > 0$ *such that*

$$[\tilde{q}_0(t)](u) \le e^{c|t-s|}[\tilde{q}_0(s)](u), \qquad u \in \mathcal{Y}, \quad t, s \in I. \tag{2.91}$$

(3) *The gauge transform* $G(t)$ *maps* $\mathcal{Y}$ *onto* $\mathcal{Y}$ *and*

$$[\tilde{q}_0(t)](G(t)u) = [q_0(t)](u), \ \ [\tilde{q}_1(t)](G(t)u) = [q_1(t)](u), \ \ u \in \mathcal{Y}. \tag{2.92}$$

Proof. By virtue of (2.61) corresponding to $\tilde{A}(t,x)$ and $\tilde{V}(t,x)$, we have

$$\|Qu\|^2 + \|\nabla_{\tilde{A}(t)}u\|^2 \le C\tilde{q}_0(t)(u), \quad u \in C_0^\infty(\mathbb{R}^d), \quad t \in I. \tag{2.93}$$

Hence, $\|\dot{\tilde{A}}(t)u\|^2 \le C\|Qu\|^2 \le C\tilde{q}_0(s)(u)$ for any $t, s \in I$ and by integration

$$\|(\tilde{A}(t) - \tilde{A}(s))u\| \le C|t-s|\tilde{q}_0(s)(u)^{\frac{1}{2}}. \tag{2.94}$$

Likewise, using, in addition to (2.93), assumption (2.25) and obvious identity $\||\dot{\tilde{V}}(r)|^{1/2}u\| = \||\dot{\tilde{V}}(r)|^{1/2}|u|\|$, we obtain that

$$\||\dot{\tilde{V}}(r)|^{1/2}u\|^2 \le C(\|\nabla|u|\|^2 + \|Qu\|^2) \le C(\|\nabla_{\tilde{A}(s)}u\|^2 + \|Qu\|^2) \le C\tilde{q}_0(s)(u).$$

Applying this to $\tilde{V}(t,x) - \tilde{V}(s,x) = \int_s^t \dot{\tilde{V}}(r,x)dr$, we have

$$|((\tilde{V}(t) - \tilde{V}(s))u, v)| \le C|t-s|\tilde{q}_0(s)(u)^{\frac{1}{2}}\tilde{q}_0(s)(v)^{\frac{1}{2}}. \tag{2.95}$$

Write $\tilde{q}_0(t)(u,v) - \tilde{q}_0(s)(u,v)$ for $u, v \in C_0^\infty(\mathbb{R}^d)$ in the form

$$\begin{aligned}(\nabla_{\tilde{A}(s)}u, i(\tilde{A}(s) - \tilde{A}(t))v) + (i(\tilde{A}(s) - \tilde{A}(t))u, \nabla_{\tilde{A}(s)}v) \\ + ((\tilde{A}(t) - \tilde{A}(s))u, (\tilde{A}(t) - \tilde{A}(s))v) + ((\tilde{V}(t) - \tilde{V}(s))u, v).\end{aligned}$$

We estimate each term separately by using (2.93), (2.94) and (2.95). We obtain for $|t-s| \le 1$ that

$$|\tilde{q}_0(t)(u,v) - \tilde{q}_0(s)(u,v)| \le C|t-s|\tilde{q}_0(s)(u)^{\frac{1}{2}}\tilde{q}_0(s)(v)^{\frac{1}{2}}. \tag{2.96}$$

It follows that $D([\tilde{q}_0(t)]) = D([\tilde{q}_0(s)])$ as in the proof of Lemma 2.5.1, all estimate above extend to u, v in $D([\tilde{q}_0(t)]) = D([\tilde{q}_0(s)])$ and

$$[\tilde{q}_0(t)](u) \le (1 + C|t-s|)[\tilde{q}_0(s)](u) \le e^{C|t-s|}[\tilde{q}_0(s)](u). \tag{2.97}$$

Argument above applies to $q_0(t)$ as well and we have (2.93) for $u \in D([q_0(t)])$; $D([q_0(t)]) = D([q_0(s)])$ for $t, s \in I$; and estimate (2.97) holds for $[q_0(t)]$ and $[q_0(s)]$. Moreover, we have $D([q_1(t)]) = D([\tilde{q}_1(t)])$ by virtue of characterization formula (2.19) of domains of the forms. Since $\|L_Q^{\frac{1}{2}}u\|^2 \le C(\|Qu\|^2 + \|\nabla_{\tilde{A}(t)}u\|^2)$ for $u \in C_0^\infty(\mathbb{R}^d)$, we also have $D([\tilde{q}_0(t)]) \subset D(L_Q^{\frac{1}{2}})$ from (2.93). Statements (1) and (2) follow.

Both $\|\nabla_{\tilde{A}(t)}G(t)u\| = \|\nabla_{A(t)}u\|$ and $(V(t)G(t)u, G(t)u) = (V(t)u, u)$ are obvious for $u \in C_0^\infty(\mathbb{R}^d)$. Since the latter space is a core of the forms $[q_0(t)]$ and $[\tilde{q}_0(t)]$, we see that $D([\tilde{q}_0(t)]) = G(t)D([q_0(t)])$, $G(t)$ maps $\mathcal{Y}$ onto $\mathcal{Y}$, and that $[\tilde{q}_0(t)](G(t)u) = [q_0(t)](u)$ for $u \in \mathcal{Y}$. The corresponding relation for $[q_1(t)]$ and $[\tilde{q}_1(t)]$ may be proved similarly. □

Before proceeding to the proof Theorem 2.1.10, we recall the following general fact: If H is a positive selfadjoint operator in a Hilbert space $\mathcal{H}$, $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ is the scale of Hilbert spaces associated with H, viz. $\mathcal{H}_1 = D(H^{1/2})$ and $\mathcal{H}_{-1} = \mathcal{H}_1^*$ with $\mathcal{H}^*$ being identified with $\mathcal{H}$, then:

(i) $\mathcal{H}_{-1}$ is the completion of $\mathcal{H}$ by the norm $\|H^{-1/2}u\|$.

(ii) H has a natural extension H_- to $\mathcal{H}_{-1}$ and H_- is selfadjoint in $\mathcal{H}_{-1}$ with domain $D(H^{1/2})$.

(iii) The part H_+ of H_- in $\mathcal{H}_1$ is again selfadjoint with domain $D(H^{3/2})$.

These should be obvious if, by using spectral representation theorem, we represent H as a multiplication operator by a positive function on $L^2(M, d\mu)$, $(M, d\mu)$ being a suitable measure space.
Proof of Theorem 2.1.10. We equip $\mathcal{Y}$ with the inner product $q_0(u, v)$ and let $\mathcal{X}$ be its dual space as in the theorem. It is obvious that $\mathcal{Y} \subset \mathcal{X}$ densely and continuously. Lemma 2.6.1 yields statement (a) except for the fact that $H(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ and it is norm continuous. To prove the latter fact, we first show that the multiplication by $\langle x\rangle^2$ is bounded from $\mathcal{Y}$ to $\mathcal{X}$ by using (2.93) for $q_0(t)$:

$$\begin{aligned} \|\langle x\rangle^2 u\|_{\mathcal{X}} &= \sup_{v\in\mathcal{Y}, \|v\|_{\mathcal{Y}}=1} |(\langle x\rangle^2 u, v)| \le C \sup_{v\in\mathcal{Y}, \|v\|_{\mathcal{Y}}=1} \|Qu\|\|Qv\| \\ &\le C \sup_{v\in\mathcal{Y}, \|v\|_{\mathcal{Y}}=1} [q_0(t_0)](u)^{\frac{1}{2}}[q_0(t_0)](v)^{\frac{1}{2}} = C\|u\|_{\mathcal{Y}}. \quad (2.98) \end{aligned}$$

Then, we estimate for $u, v \in C_0^\infty(\mathbb{R}^d)$ via (2.92) for $[q_0(t)]$ as follows:

$$|(H(t)u, v)| \le |q_0(t)(u, v)| + |(C(t)\langle x\rangle^2 u, v)| \le C(e^{2c|t-t_0|} + C(t))\|u\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}}.$$

and $\|H(t)u\|_{\mathcal{X}} \le C\|u\|_{\mathcal{Y}}$. This extends to $u \in \mathcal{Y}$ since $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{Y}$. Thus, $H(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$. We have

$$\begin{aligned} ((H(t) - H(s))u, v) &= ((H_0(t) - H_0(s))u, v) - ((C(t) - C(s))\langle x\rangle^2 u, v) \\ &= ((q_0(t) - q_0(s))u, v) - ((C(t) - C(s))\langle x\rangle^2 u, v), \quad u, v \in \mathcal{Y}. \end{aligned}$$

Thus, (2.96) for $q_0(t)$ and (2.98) imply $\|H(t) - H(s)\|_{\mathbf{B}(\mathcal{Y},\mathcal{X})} \le C(|t-s| + |C(t) - C(s)|)$ and statement (a) follows.

We define $\mathcal{Y}_t$ to be $\mathcal{Y}$ with new inner product $(u, v)_{\mathcal{Y}_t} = [\tilde{q}_0(t)](u, v)$ and $\mathcal{X}_t$ to be the dual space of $\mathcal{Y}_t$ with respect to the inner product of $\mathcal{H}$. Then, $\mathcal{X}_t \subset \mathcal{H} \subset \mathcal{Y}_t$ is the scale of Hilbert space associated with positive selfadjoint operator $\tilde{H}_0(t)$. Then, by virtue of the property (i), $\mathcal{X}_t$ is independent of

t as a set and is equal to $\mathcal{X}$ since $\mathcal{Y}_t = \mathcal{Y}$ is independent of t as a set with equivalent Hilbert space structures. Properties (ii) and (iii) produce selfadjoint operators $\tilde{H}_0(t)_-$ and $\tilde{H}_0(t)_+$ in $\mathcal{X}_t$ and $\mathcal{Y}_t$ respectively. It is evident that $\tilde{H}_0(t)_-$ is a closed operator in $\mathcal{X}$ (with respect to the original norm) and $\tilde{H}_0(t)_+$ is its part in $\mathcal{Y}$. We now want to apply Theorem 2.2.1 to triplet $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$.

We check conditions of Theorem 2.2.1 for $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$. Norm $\|u\|_{\mathcal{Y}_t}$ is equivalent with the original one of $\mathcal{Y}$ by virtue of the closed graph theorem. Estimate (2.91) implies that $\{\|u\|_{\mathcal{Y}_t} : t \in I\}$ satisfies condition (2.27) of Theorem 2.2.1 for $\mathcal{Y}_t$ and likewise for $\mathcal{X}_t$ by duality. From (2.96) we have

$$|\langle(\tilde{H}_0(t)_- - \tilde{H}_0(s)_-)u, v\rangle| \leq c|t-s|\tilde{q}_0(s)(u)^{\frac{1}{2}}\tilde{q}_0(s)(v)^{\frac{1}{2}}, \tag{2.99}$$

where $\langle\cdot,\cdot\rangle$ on the left is the coupling between $\mathcal{X}$ and $\mathcal{Y}$. This implies that

$$\|(\tilde{H}_0(t)_- - \tilde{H}_0(s)_-)u\|_{\mathcal{X}_s} \leq c|t-s|\|u\|_{\mathcal{Y}_s} \tag{2.100}$$

and we see that $I \ni t \to \tilde{H}_0(t)_- \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ is norm continuous.

Thus, there uniquely exists a family of operators $\{\tilde{U}(t,s) : t, s \in I\}$ which satisfies the properties of Theorem 2.2.1 for $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$. We define

$$U(t,s) = G(t)^{-1}\tilde{U}(t,s)G(s).$$

We know that $G(t)$ maps $\mathcal{Y}$ onto $\mathcal{Y}$ by virtue of Lemma 2.6.1 and, (2.98) implies that, for $u \in \mathcal{Y}$, $I \ni t \mapsto G(t)u \in \mathcal{X}$ is continuously differentiable. Then, it is easy to check that $U(t,s)$ is satisfies all properties of statement (b) except that $U(t,s)$ is a strongly continuous family of unitary operators in $\mathcal{H}$, which we now show. Define $u(t) = U(t,s)\varphi$ for $\varphi \in \mathcal{Y}$. Then, with $\langle\cdot,\cdot\rangle$ being the coupling of $\mathcal{X}$ and $\mathcal{Y}$, we have

$$\begin{aligned}\partial_t(u(t), u(t))_{L^2} &= 2\Re\langle -iH(t)u(t), u(t)\rangle \\ &= 2\Re\{-iq_0(t)(u(t), u(t)) + iC(t)\langle\langle x\rangle^2 u(t), u(t)\rangle\} = 0.\end{aligned}$$

It follows that $\|u(t)\| = \|\varphi\|$ and, since $\mathcal{Y}$ is dense in $\mathcal{H}$, we conclude $U(t,s)\mathcal{H} \subset \mathcal{H}$ and $\|U(t,s)\varphi\| = \|\varphi\|$ for all $\varphi \in \mathcal{H}$. Then, $U(t,s)$ must be unitary since $U(t,s)U(s,t)\varphi = \varphi$. If $\varphi \in \mathcal{Y}$, $(t,s) \mapsto U(t,s)\varphi \in \mathcal{H}$ is continuous in $\mathcal{H}$. Hence $U(t,s)$ is strongly continuous in $\mathbf{B}(\mathcal{H})$ by the unitarity. The uniqueness of $U(t,s)$ of Theorem 2.1.10 follows from the uniqueness result of Theorem 2.2.1 by tracing back the argument above. □

Chapter 3

Absence of zero resonances of massless Dirac operators.

3.1 Introduction, assumption and theorems.

We consider the massless Dirac operator

$$H = \alpha \cdot D + Q(x), \ D = -i\nabla_x, \ x \in \mathbb{R}^3, \tag{3.1}$$

acting on $\mathbb{C}^4$-valued functions on $\mathbb{R}^3$. Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices:

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \ j = 1, 2, 3,$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we use the vector notation that $(\alpha \cdot D)u = \sum_{j=1}^{3} \alpha_j(-i\partial_{x_j})u$. We assume that $Q(x)$ is a 4×4 Hermitian matrix valued function satisfying the following assumption:

Assumption 3.1.1. *There exists positive constant C and $\rho > 1$ such that, for each component $q_{jk}(x)$ $(j, k = 1, \cdots, 4)$ of $Q(x)$,*

$$|q_{jk}(x)| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^3.$$

where $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$.

We remark that the Dirac operator for a Dirac particle minimally coupled to the electromagnetic field described by the potential (q, A) is given by

$$\alpha \cdot (D - A(x)) + q(x)I_4, \tag{3.2}$$

where I_4 is the 4×4 identity matrix, and is a special case of (3.1.1). And, if $q(x) = 0$, (3.2) reduces to

$$\alpha \cdot (D - A(x)) = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A(x)) \\ \sigma \cdot (D - A(x)) & \mathbf{0} \end{pmatrix},$$

where $\sigma \cdot (D - A(x))$ is the Weyl-Dirac operator.

To state the result of the paper, we introduce some notation and terminology. $\mathcal{F}$ is the Fourier transform:

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\xi) e^{-ix\cdot\xi} d\xi.$$

We often write $\hat{f}(\xi) = (\mathcal{F}f)(\xi)$ and $\check{f}(\xi) = (\mathcal{F}^{-1}f)(\xi)$. $\mathcal{L}^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the Hilbert space of all $\mathbb{C}^4$-valued square integrable functions. For $s \in \mathbb{R}$, $\mathcal{L}^{2,s}(\mathbb{R}^3) = L^{2,s}(\mathbb{R}^3, \mathbb{C}^4) := \langle x \rangle^{-s} L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the weighted $\mathcal{L}^2(\mathbb{R}^3)$ space. $\mathcal{S}'(\mathbb{R}^3) = \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ is the space of $\mathbb{C}^4$-valued tempered distributions. $\mathcal{H}^s(\mathbb{R}^3) = H^s(\mathbb{R}^3, \mathbb{C}^4)$ is the Sobolev space of order s:

$$\mathcal{H}^s(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3) | \hat{f} \in \mathcal{L}^{2,s}(\mathbb{R}^3)\}$$

with the inner product $(f, g)_{\mathcal{H}^s} = \sum_{j=1}^4 (\hat{f}_j, \hat{g}_j)_{L^{2,s}}$. The spaces $\mathcal{H}^{-s}(\mathbb{R}^3)$ and $\mathcal{H}^s(\mathbb{R}^3)$ are duals of each other with respect to the coupling

$$\langle f, g \rangle := \sum_{j=1}^4 \int_{\mathbb{R}^3} (\mathcal{F}f_j)(\xi) \overline{(\mathcal{F}g_j(\xi))} d\xi, \quad f \in \mathcal{H}^{-s}(\mathbb{R}^3), g \in \mathcal{H}^s(\mathbb{R}^3).$$

For Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, $B(\mathcal{X}, \mathcal{Y})$ stands for the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$, $B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$.

It is well known that the free Dirac operator $H_0 := \alpha \cdot D$ is self-adjoint in $\mathcal{L}^2(\mathbb{R}^3)$ with domain $\mathcal{D}(H_0) = \mathcal{H}^1(\mathbb{R}^3)$. Hence by the Kato-Rellich theorem, H is also self-adjoint in $\mathcal{L}^2(\mathbb{R}^3)$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. We denote their self-adjoint realizations again by H_0 and H respectively. In what follows, we write $H_0 f$ also for $(\alpha \cdot D)f$ when $f \in \mathcal{S}'(\mathbb{R}^3)$.

Definition 3.1.2. *If $f \in \mathcal{L}^2(\mathbb{R}^3)$ satisfies $Hf = 0$, we say f is a zero mode of H. If $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ satisfies $Hf = 0$ in the sense of distributions, but $f \notin \mathcal{L}^2(\mathbb{R}^3)$, then f is said to be a zero resonance state and zero is a resonance of it.*

The following is the main result of this paper.

Theorem 3.1.3. *Let $Q(x)$ satisfy Assumption 3.1.1. Suppose $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ satisfies $Hf = 0$ in the sense of distributions, then for any $\mu < 1/2$, we have $\langle x\rangle^{\mu} f \in \mathcal{H}^1(\mathbb{R}^3)$. In particular, there are no resonance for H.*

Remark 3.1.4. *The decay result $\langle x\rangle^{\mu} f \in \mathcal{H}^1(\mathbb{R}^3)$, $\mu < 1/2$ cannot be improved. This can be seen from the example of zero mode of the Weyl-Dirac operator which was constructed by Loss-Yau [17]. Loss and Yau have constructed a vector potential $A_{LY}(x)$ and a zero mode $\phi_{LY}(x)$ satisfying $\sigma \cdot (D - A_{LY}(x))\phi_{LY} = 0$, where A_{LY} and ϕ_{LY} satisfy $A_{LY}(x) = \mathcal{O}(\langle x\rangle^{-2})$, $|\phi_{LY}(x)| = \langle x\rangle^{-2}$. Define $f_{LY} = {}^t(0, \phi_{LY})$ and $Q(x) = -\alpha \cdot A_{LY}(x)$, then*

$$Hf_{LY} = (H_0 + Q)f_{LY} = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A_{LY}(x)) \\ \sigma \cdot (D - A_{LY}(x)) & \mathbf{0} \end{pmatrix} f_{LY} = 0,$$

and $f_{LY} \in \mathcal{L}^{2,\mu}(\mathbb{R}^3)$ for any $\mu < 1/2$. However, $f_{LY} \notin \mathcal{L}^{2,\frac{1}{2}}(\mathbb{R}^3)$.

We remark that Saitō - Umeda [21] and Zhong - Gao [30] have proven the following result under the same assumption $|Q(x)| \le C\langle x\rangle^{-\rho}$, $\rho > 1$ (In [21], it is assumed $\rho > 3/2$, however, arguments of [21] go through under the assumption $\rho > 1$ as was made explicit in [30]): If f satisfies $f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ for some $0 < s \le \min\{3/2, \rho - 1\}$ and $Hf = 0$ in the sense of distributions, then $f \in \mathcal{H}^1(\mathbb{R}^3)$. Our theorem improves over the results of [21] and [30] by weakening the assumption $f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ to $\mathcal{L}^{2,-3/2}(\mathbb{R}^3)$, which is $\rho > 1$ independent, and by strengthening the result $f \in \mathcal{H}^1(\mathbb{R}^3)$ to a sharp decay estimate $\langle x\rangle^{\mu} f \in \mathcal{H}^1(\mathbb{R}^3)$, $\mu < 1/2$. We briefly explain the significance of the theorem.

The solution of the time-dependent Dirac equation

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = \phi$$

is given by $e^{-itH}\phi$. Under Assumption 3.1.1, it has been proven that the spectrum $\sigma(H) = \mathbb{R}$, the limiting absorption principle is satisfied and that $\sigma_p(H)\backslash\{0\}$ is discrete. To make the argument simple, we assume $\sigma_p(H) \subset \{0\}$. Then for $\phi \in \mathcal{L}^2_{ac}(H)$, the absolutely continuous spectral subspace of $\mathcal{L}^2$ for H, $e^{-itH}\phi$ may be represented in terms of the boundary values of the resolvent $(H - \lambda \pm i0)^{-1}$:

$$e^{-itH}\phi = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} e^{-it\lambda}\{(H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1}\}\phi d\lambda, \ t > 0,$$

and the asymptotic behavior as $t \to \pm\infty$ of $e^{-itH}\phi$ depends on whether (1) $\lambda = 0$ is a regular point, viz, $(H - (\lambda \pm i0))^{-1}$ is smooth up to $\lambda = 0$, (2)

$\lambda = 0$ is a resonance of it, (3) $\lambda = 0$ is an eigenvalue of H or (4) $\lambda = 0$ is an eigenvalue at the same time is a resonance. Thus, Theorem 3.1.3 eliminates the possibility (2) and (4). We should recall that if $m \neq 0$, then all four cases mentioned above appear at the threshold points $\pm m$. It is well-known that $\lambda = 0$ is not a regular point if $f + (H_0 \pm i0)^{-1} Q f = 0$ has a non-trivial solution $f \in \mathcal{L}^{2,-\rho/2}$ and this f satisfies $Hf = 0$. Note that, to conclude that $f \in \mathcal{H}^1$ by using results of [21] or [30], we need assume $0 < \rho/2 \leq \min\{3/2, \rho - 1\}$ or $2 \leq \rho \leq 3$, which is a severe restriction for this application, whereas Theorem 3.1.3 does not impose only such restriction.

The rest of the paper is devoted to the proof of Theorem 3.1.3. In section 2, we prepare some lemmas for proving the main theorem. In section 3, we prove the main theorem 3.1.3.

3.2 Preliminaries.

In this section, we prepare some lemmas which are necessary for proving the theorem. We use the following well-known lemma:

Theorem 3.2.1. (Nirenberg - Walker [19]) Let $1 < p < \infty$ and let $a, b \in \mathbb{R}$ be such that $a + b > 0$. Define

$$k(x,y) = \frac{1}{|x|^a |x-y|^{d-(a+b)} |y|^b}, \quad x, y \in \mathbb{R}^d, \ x \neq y.$$

Then, integral operator

$$(K\phi)(x) = \int_{\mathbb{R}^d} k(x,y)\phi(y)dy$$

is bounded in $L^p(\mathbb{R}^d)$ if and only if $a < d/p$ and $b < d/q$, where $q = p/(p-1)$ is the dual exponent of p.

For $f = {}^t(f_1, f_2, f_3, f_4)$, we define the integral operator A by

$$(Af)(x) = \frac{i}{4\pi} \int_{\mathbb{R}^3} \frac{\alpha \cdot (x-y)}{|x-y|^3} f(y)dy.$$

Since

$$\frac{i}{4\pi} \mathcal{F}^{-1}\left(\frac{\xi}{|\xi|^3}\right)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{x}{|x|^2},$$

it is obvious that

$$\mathcal{F}^{-1}(Af)(x) = \frac{\alpha \cdot x}{|x|^2} (\mathcal{F}^{-1} f)(x) = (\alpha \cdot x)^{-1} (\mathcal{F}^{-1} f)(x) \tag{3.3}$$

Lemma 3.2.2. *For any $t \in (-\frac{3}{2}, \frac{1}{2})$, $A \in B(\mathcal{L}^{2,-t}(\mathbb{R}^3), \mathcal{L}^{2,-t-1}(\mathbb{R}^3))$.*

Proof. The multiplication by $\langle x\rangle^t$ is isomorphism from $\mathcal{L}^2(\mathbb{R}^3)$ onto $\mathcal{L}^{2,-t}(\mathbb{R}^3)$. It follows that $A \in B(\mathcal{L}^{2,-t}(\mathbb{R}^3), \mathcal{L}^{2,-t-1}(\mathbb{R}^3))$ if and only if $\langle x\rangle^{-t-1}A\langle x\rangle^t \in B(\mathcal{L}^2(\mathbb{R}^3))$. The integral kernel of $\langle x\rangle^{-t-1}A\langle x\rangle^t$ is bounded by

$$\frac{1}{4\pi\langle x\rangle^{t+1}|x-y|^2\langle y\rangle^{-t}}.$$

Lemma 3.2.2 follows by applying Lemma 3.2.1 with $a = t+1,\ b = -t,\ d = 3,\ p = q = 2$. □

Lemma 3.2.3. *Let $-3/2 < s < 1/2$. Then for any $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ and $\phi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$, we have the identity;*

$$\langle \mathcal{F}^{-1}(Ag), \phi\rangle = \langle \mathcal{F}^{-1}g, \frac{\alpha\cdot x}{|x|^2}\phi\rangle. \tag{3.4}$$

Proof. We note that both $\mathcal{F}^{-1}g \in \mathcal{H}^{-s-1}(\mathbb{R}^3)$ and $\mathcal{F}^{-1}(Ag) \in \mathcal{H}^{-s-1}(\mathbb{R}^3)$. Indeed, the former is obvious by $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3) \subset \mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ and the latter follows since $Ag \in \mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ by virtue of the assumption $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$, $-\frac{3}{2} < s < \frac{1}{2}$ and Lemma 3.2.2. Let $\phi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$. Take a sequence $g_n \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that $\lim_{n\to\infty}\|g_n - g\|_{\mathcal{L}^{2,-s}} = 0$. Since A is continuous from $\mathcal{L}^{2,-s}(\mathbb{R}^3)$ to $\mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ by virtue of Lemma 3.2.2, it follows that

$$\begin{aligned}\langle \mathcal{F}^{-1}(Ag), \phi\rangle &= \lim_{n\to\infty}\langle \mathcal{F}^{-1}(Ag_n), \phi\rangle \\ &= \lim_{n\to\infty}\langle \frac{\alpha\cdot x}{|x|^2}\mathcal{F}^{-1}g_n, \phi\rangle \\ &= \lim_{n\to\infty}\langle \mathcal{F}^{-1}g_n, \frac{\alpha\cdot x}{|x|^2}\phi\rangle \\ &= \langle \mathcal{F}^{-1}g, \frac{\alpha\cdot x}{|x|^2}\phi\rangle.\end{aligned}$$

Here we used (3.3) in the second step and that $\frac{\alpha\cdot x}{|x|^2}\phi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$ in the final step. This completes the proof. □

The following is an extension of Theorem 4.1 of [21] and plays an important role in the proof of theorem.

Lemma 3.2.4. *Suppose that $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $H_0 f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ for some $s \in (-\frac{3}{2}, \frac{1}{2})$. Then, f satisfies $AH_0 f = f$.*

Proof. Since $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $AH_0 f \in \mathcal{L}^{-s-1}(\mathbb{R}^3) \subset \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ by virtue of Lemma 3.2.2, it follows that $\mathcal{F}^{-1}f, \mathcal{F}^{-1}(AH_0 f) \in \mathcal{H}^{-3/2}(\mathbb{R}^3)$. Thus, it suffice to show that

$$\langle \mathcal{F}^{-1}(AH_0 f), \phi\rangle = \langle \mathcal{F}^{-1}f, \phi\rangle, \ for\ any\ \phi \in \mathcal{H}^{3/2}(\mathbb{R}^3). \tag{3.5}$$

Since $C_0^\infty(\mathbb{R}^d\backslash\{0\}, \mathbb{C}^4)$ is dense in $\mathcal{H}^s(\mathbb{R}^d)$ for $s \le d/2$, we need only prove (3.5) for $\phi \in C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$. By setting $g = H_0 f$ in (3.4) and using $\mathcal{F}^{-1}(H_0 f)(x) = (\alpha\cdot x)(\mathcal{F}^{-1}f)(x)$ for $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle \mathcal{F}^{-1}(AH_0 f), \phi\rangle &= \langle (\alpha\cdot x)\mathcal{F}^{-1}f, \frac{\alpha\cdot x}{|x|^2}\phi\rangle \\ &= \langle \mathcal{F}^{-1}f, \frac{(\alpha\cdot x)^2}{|x|^2}\phi\rangle = \langle \mathcal{F}^{-1}f, \phi\rangle.\end{aligned}$$

This completes the proof. □

3.3 Proof of Theorem 3.1.3

We may assume $1 < \rho < 3$ without losing generality. We apply well-known Agmon's bootstrap argument. Let $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $Hf = 0$ in the sense of distributions. Then $H_0 f = -Qf \in \mathcal{L}^{2,-\frac{3}{2}+\rho}(\mathbb{R}^3)$ by the assumption 3.1.1. Since $-\frac{1}{2} < \rho - \frac{3}{2} < \frac{3}{2}$, we have $AQf \in \mathcal{L}^{2,-\frac{3}{2}+\rho-1}(\mathbb{R}^3)$ by virtue of Lemma 3.2.2. Then Lemma 3.2.4 implies $f = AH_0 f = -AQf \in \mathcal{L}^{2,-\frac{3}{2}+\rho-1}(\mathbb{R}^3)$. Thus we may repeat the argument several times and obtain $f \in \mathcal{L}^{2,-\frac{3}{2}+n(\rho-1)}(\mathbb{R}^3)$ as long as $-\frac{3}{2}+n(\rho-1)+1 < \frac{3}{2}$. Let n_0 be the largest integer such that $-\frac{3}{2}+n_0(\rho-1)+1 < \frac{3}{2}$ so that $f \in \mathcal{L}^{2,-\frac{3}{2}+n_0(\rho-1)}(\mathbb{R}^3)$ and $Qf \in \mathcal{L}^{2,-\frac{3}{2}+n_0(\rho-1)+\rho}(\mathbb{R}^3)$, however $-\frac{3}{2}+n_0(\rho-1)+\rho > \frac{3}{2}$. Then for $\mu < 1/2$ arbitrary close to 1/2, $H_0 f = -Qf \in \mathcal{L}^{2,\mu+1}(\mathbb{R}^3)$. Thus, $f \in \mathcal{L}^{2,\mu}(\mathbb{R}^3)$ by virtue of Lemma 3.2.2 and Lemma 3.2.4. By differentiating, we have

$$\begin{aligned}H_0\langle x\rangle^\mu f &= -i\mu(\alpha\cdot x)\langle x\rangle^{\mu-2}f + \langle x\rangle^\mu H_0 f \\ &= -i\mu(\alpha\cdot x)\langle x\rangle^{\mu-2}f - \langle x\rangle^\mu Qf \in \mathcal{L}^2(\mathbb{R}^3).\end{aligned}$$

It follows that $\mathcal{F}(\langle x\rangle^\mu f) \in \mathcal{L}^{2,1}(\mathbb{R}^3)$ which is equivalent to $\langle x\rangle^\mu f \in \mathcal{H}^1(\mathbb{R}^3)$. This completes the proof of Theorem 3.1.3.

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