ON CONVEX UNIVALENT FUNCTIONS

-ABSTRACT-

T.CHIBA

Let F denote the class of functions which are regular and convex for $|z|< 1$ and are normalized by $f(0)=0$ and $f'(0)=1$. T.Basgöze, J.L.Frank and F.R.Keogh [1] showed the solution of the problem of determing necessary and sufficient conditions on complex numbers λ,μ under which for all $f(z)=z+\sum_{2} a_n z^n \in F$, $\lambda z+\mu a_2 z^2$ is convex and

$$\frac{1}{2}z \prec \lambda z+\mu a_2 z^2 \prec f(z).$$

On the basis of those ideas we consider the conditions of complex numbers λ,μ under which $\lambda z+\mu a_n z^n$ is convex and

$$\frac{1}{2}z \prec \lambda z+\mu a_n z^n \prec f(z).$$

Theorem 3 gives the necessary conditions for λ,μ. In §.3 we discuss the sufficiency of λ,μ which is described by Theorem 3. However, in general, it is difficult to consider them for all $f(z)\in F$, and $n\in\mathbb{N}$, because it needs the explicit solutions of the algebraic equations of degree n. But in Theorem 5 for each odd number n, we show a result. Furthermore, Theorem 6 shows that if n is odd, the conditions of λ,μ which we have obtaind in Theorem 3 is necessary and sufficient with respect to the representative convex function $f(z)=\dfrac{z}{1-z}$.

[1] T.Basgöze, J.L.Frank and F.R.Keogh, On convex univalent functions, Can. J.Math.22 (1970)

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§.1 Introduction

Let F denote the class of functions which are regular and convex for $|z|< 1$ and are normalized by

$$f(0)=0, \ f'(0)=1.$$

T.Basgöze, J.L.Frank and F.R.Keogh [1] showed the solution of the problem of determing necessary and sufficient conditions on complex numbers λ, μ under which, for all $f(z)= z+\sum_2 a_n z^n \in F$,

$\lambda z+\mu a_2 z^2$ is convex and

$$\frac{1}{2} z \prec \lambda z+\mu a_2 z^2 \prec f(z) .$$

On the basis of those ideas we consider the conditions of complex numbers λ, μ under which

$\lambda z+\mu a_n z^n$ is convex and

$$\frac{1}{2} z \prec \lambda z+\mu a_n z^n \prec f(z) .$$

§.2 Suppose that f(z) and g(z) are regular for $|z|< 1$, f(z) is univalent, and g(z) is subordinate to f(z), that is,

$$g(0)=f(0)=0, \quad g(|z|<1) \subset f(|z|<1).$$

We shall then write

$$g(z) \prec f(z).$$

For all $w = f(z) \in F$, it is known that $f(|z|<1)$ contains the disk $|w| < \frac{1}{2}$, i.e., that

$$\frac{1}{2} z \prec f(z), \text{ (see, for example, [2] P.13).}$$

The purpose of this note is to consider the conditions of complex numbers λ, μ under which, for all $f(z) \in F$, $\lambda z + \mu a_n z^n$ is convex and

$$(*) \quad \frac{1}{2} z \prec \lambda z + \mu a_n z^n \prec f(z).$$

The solution for $n=2$ was showed by T.Basgöze, J.L.Frank and F.R.Keogh [1] .
In this case the relation between λ, μ is

$$\lambda = \mu + \frac{1}{2} \quad , \; 0 \leqq \mu \leqq \frac{1}{6} \; .$$

Theorem 1. If the left-hand relation in $(*)$ is satisfied for all $f(z) = z + \sum_2 a_n z^n \in F$, then

$$|\lambda| \geqq \frac{1}{2} \; .$$

Proof. Generally $g(z) \prec f(z)$ implies $|g'(0)| \leqq |f'(0)|$ (see, for example, [3] P.228). Q.E.D.

By considering $\lambda e^{i\theta} z + \mu e^{in\theta} a_n z^n$, it is easily seen that we may suppose with no effective loss of generality that λ is real and positive.

Theorem 2. The function $z + c z^n$ is convex if and only if $|c| \leqq \frac{1}{n^2}$.

Proof. For all $f(z) \in F$, it is known that the function $zf'(z)$ is starlike for $|z| < 1$ [4] .

If $z + c\, z^n \in F$, then $zf'(z) = z + n \cdot c \cdot z^n$, this implies

$$n \cdot n\,|c| \leq 1\ ,\ \text{i.e.,}\ |c| \leq \frac{1}{n^2}\ [5]\ .$$

Conversely, if $|c| \leq \frac{1}{n^2}$, then

$$\left|\mathrm{Re}\left(z\frac{f''(z)}{f'(z)}\right)\right| \leq \left|z\frac{f''(z)}{f'(z)}\right| \leq n(n-1)\left|\frac{cz^{n-1}}{1-ncz^{n-1}}\right| \leq 1.$$

So that $z + c\, z^n$ is convex. Q.E.D.

Theorem 3. If for all $f(z) \in F$, $\lambda z + \mu\, a_n z^n$ is convex and $\frac{1}{2}z \prec \lambda z + \mu\, a_n z^n \prec f(z)$, then

$$\mu = (-1)^n\left(\lambda - \frac{1}{2}\right),$$
$$0 \leq (-1)^n \mu \leq \frac{1}{2(n^2-1)}\quad (n = 2, 3, \cdots).$$

Proof. If $\lambda z + \mu a_n z^n$ is convex for all $f(z) \in F$, then, with $f(z) = \dfrac{z}{1-z} = z + z^2 + \cdots$,

by Theorem 2 we must have $\left|\frac{\mu}{\lambda}\right| \leq \frac{1}{n^2}$, i.e.,

$$|\mu| \leq \frac{1}{n^2}\lambda\ \cdots\cdots\cdots\ (1).$$

The minimum value of $|\lambda z + \mu z^n|$ on $|z| = 1$ is then $\lambda - |\mu|$; hence $\frac{1}{2}z \prec \lambda z + \mu z^n$ implies

$$\frac{1}{2} \leq \lambda - |\mu|\ \cdots\cdots\ (2).$$

With the same $f(z)$, if $\lambda z + \mu z^n \prec f(z)$, then, for all real x , $-1 < x < 1$, we have

$$-\frac{1}{2} < \lambda x + \mathrm{Re}\,\mu \cdot x^n,$$

and allowing $x \to -1$,

$$\lambda - \frac{1}{2} = (-1)^n\, \mathrm{Re}\,\mu\ \cdots\cdots\cdots\ (3).$$

If we put $\lambda - \frac{1}{2} = \nu\ (\geq 0)$, $\mu = a + ib\ (a, b \in \mathbb{R})$, then from (3) we have

$$(-1)^n a \geq \nu\ ,\ \text{i.e.,}\ a^2 \geq \nu^2\ \cdots\cdots\cdots\ (4).$$

And from (2) we have

$$\nu^2 \geq a^2 + b^2\ \cdots\cdots\cdots\ (5).$$

From (4), (5) we obtain $b = 0$, $\nu = (-1)^n a$.

This implies

$$a = (-1)^n \nu, \text{ or}$$
$$\mu = (-1)^n (\lambda - \frac{1}{2}).$$

Further from $(-1)^n \mu = \lambda - \frac{1}{2} \geqq 0$, $(-1)^n \mu = |\mu|$, we get by (1)

$$(-1)^n \mu \leqq \frac{1}{n^2}\left\{(-1)^n \mu + \frac{1}{2}\right\}, \text{i.e.,}$$
$$0 \leqq (-1)^n \mu \leqq \frac{1}{2(n^2-1)}.$$

Q.E.D.

Theorem 4. If about λ, μ, the conditions

$$\mu = (-1)^n (\lambda - \frac{1}{2}),$$
$$0 \leqq (-1)^n \mu \leqq \frac{1}{2(n^2-1)}$$

are satisfied, then $\lambda z + \mu a_n z^n$ is convex.

Proof. Suppose that

$$\left| \frac{\mu a_n}{(-1)^n \mu + \frac{1}{2}} \right| > \frac{1}{n^2}.$$

From $(-1)^n \mu = |\mu|$, $|a_n| \leqq 1$ [4], we get

$$\frac{(-1)^n \mu}{(-1)^n \mu + \frac{1}{2}} > \frac{1}{n^2},$$

and we obtain

$$(-1)^n \mu > \frac{1}{2(n^2-1)}.$$

This is a contradiction.

And from Theorem 2, the proof is completed. Q.E.D.

§.3 The purpose of the remainder of this note is to consider the relations of $f(z) \in F$, $V_n(z,f)$, $U_n(z,f)$, where $V_n(z,f)$ and $U_n(z,f)$ are defined by

$$V_n(z,f) = \lambda z + (-1)^n (\lambda - \tfrac{1}{2}) a_n z^n \quad ,$$

$$U_n(z,f) = \left\{ \frac{1}{2} + \frac{(-1)^n}{2(n^2-1)} \right\} z + \frac{(-1)^n}{2(n^2-1)} a_n z^n \quad .$$

For $n=2$, the relation

$$V_2(z,f) \prec U_2(z,f) \prec f(z)$$

is known [1] .

But in general it is difficult to consider them for all $f(z) \in F$ and $n \in \mathbb{N}$, because it needs the explicit solutions of the algebraic equations of degree n. But for each odd number n, we get next theorem.

Theorem 5. For all $f(z) \in F$, and odd number n , we get the relation

$$U_n(z,f) \prec V_n(z,f),$$

in particular, for $\lambda \leqq \frac{1}{2} \operatorname{Max} |f(z)| + \frac{1}{4}$,

$$U_n(z,f) \prec V_n(z,f) \prec f(z), \quad (n=2m+1, m=1,2, \cdots\cdots\cdot).$$

Proof. By Theorem 2, $V_n(z,f)$ and $U_n(z,f)$ are convex. Furthermore since

$$\frac{1}{2} = \min_{|z|=1} |V_n(z,f)| \geqq \operatorname*{Max}_{|z|\leqq 1} |U_n(z,f)| = \frac{n^2-2+|a_n|}{2(n^2-1)} \quad ,$$

we have

$$U_n(|z|<1,f) \subset V_n(|z|<1,f),$$

and we can write

$$U_n(z,f) \prec V_n(z,f).$$

Since $|a_n| \leqq 1$ (see, for example, 4 P.307) , we get

$$\operatorname{Max} |V_n(z,f)| = \lambda + (\lambda - \tfrac{1}{2}) |a_n| \leqq 2\lambda - \tfrac{1}{2} .$$

Then if we assume $\lambda \leqq \frac{1}{2} \operatorname{Max} |f(z)| + \frac{1}{4}$, we get

$$V_n(|z|<1,f) \subset f(|z|<1),$$

that is, $V_n(z,f) \prec f(z)$. Thus the Theorem is proved.

Q.E.D.

Theorem 6. Let $f(z) = \frac{z}{1-z}$ for $|z|<1$. Then for each odd number n, we get the relation

$$U_n(z,f) \prec V_n(z,f) \prec f(z) \text{ .}$$

Proof. It is well known that $w=f(z) \in F$, and it maps $|z|<1$ onto the half-plane $\operatorname{Re} w > -\frac{1}{2}$.

Furthermore considering

$$V_n(1,f) = \frac{1}{2} \text{ , } V_n(-1,f) = -\frac{1}{2} \text{ , } \overline{V_n(\overline{z},f)} = V_n(z,f),$$

and univalency and convexity of $V_n(z,f)$, we get

$$V_n(z,f) \prec f(z).$$

Combining Theorem 5, the theorem is proved. Q.E.D.

In conclusion we remark that from Theorem 3 and Theorem 6, necessary and sufficient conditions on λ , μ is

$$\mu = (-1)^n (\lambda \pm \frac{1}{2}), \quad 0 \leqq (-1)^n \mu \leqq \frac{1}{2(n^2-1)}$$

with respect to $f(z) = \frac{z}{1-z}$, and odd number n.

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