A NOTE ON THE RIEMANN MAPPING THEOREM

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ABSTRACT.

The object of this note is to prove that the set of normalized conformal mappings which satisfy the Riemann Mapping Theorem is homeomorphic to a region of definition.

The Riemann Mapping Theorem [1] states: given any simply connected region $\Omega$ which is not whole plane, and a point $a \in \Omega$, there exists a unique analytic function $f_a$ in $\Omega$, normalized by the conditions $f_a(a) = 0$, $f_a'(a) > 0$, such that $f_a$ defines a one-to-one mapping of $\Omega$ onto the open unit disk $D = \{ w | |w| < 1 \}$.

We denote the set of these functions by $\mathcal{F}(\Omega)$, i.e., $\mathcal{F}(\Omega) = \{ f_a | a \in \Omega \}$. For any $f, g \in \mathcal{F}(\Omega)$, we shall define a distance $d_0(f, g)$ between these functions. For this purpose we need, first of all, an exhaustion of $\Omega$ by an increasing sequence of compact sets $K_n \subset \Omega$. By this we mean that every compact subset $K$ of $\Omega$ shall be contained in a $K_n$. Let

$$K_n = \{ z | z \in \Omega, \ |z| \leq n, \ d(z, C - \Omega) \geq 1/n \}.$$ 

Then $K_n \ (n \geq 1)$ satisfy two conditions

1. $K_n \supseteq K_{n+1}$
2. $\Omega = \bigcup_{n=1}^{\infty} K_n$.

And it is clear that each $K_n$ is bounded and closed, hence compact. Any compact set $K \subset \Omega$ is bounded and at positive distance from $\partial \Omega$; therefore it is contained in a $K_n$.

We set

$$p_n(f, g) = \sup_{z \in K_n} | f(z) - g(z) | = \max_{z \in K_n} | f(z) - g(z) |$$

for $f, g \in \mathcal{F}(\Omega)$, which may be described as the distance between $f$ and $g$ on $K_n$. 

-19-
Finally, we agree on the definition
\[
d_0(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(f, g)}{1 + p_n(f, g)}
\]

It is clear that \(d_0(f, g)\) is finite and satisfies all the conditions for a distance.

By this definition we obtain next lemma:

LEMMA 1. For sequence \(\{ f_n \}\) of \(\mathcal{F}(\Omega)\), the following two conditions are equivalent:

(*) \(\{ f_n \}\) converges to \(f\) in the sense of \(d_o\)-distance, i.e., \(\lim_{n \to \infty} d_o(f_n, f) = 0\),

(**) \(\{ f_n \}\) converges uniformly to \(f\) in the wider sense on \(\Omega\).

PROOF. (*) \(\Rightarrow\) (**):

If \(K\) be any compact set of \(\Omega\), we have \(K \subset K_n\), for large \(n\). For each \(\epsilon > 0\), if we set \(\delta > 0\), \(\frac{\delta}{1 - \delta} < \epsilon\), then we can suppose
\[
d_o(f_k, f) < \delta / 2^n,
\]
for all \(k\) greater than a certain \(K_o\), because \(d_o(f_k, f) \to 0\).

Then
\[
\frac{1}{2^n} \cdot \frac{p_n(f_k, f)}{1 + p_n(f_k, f)} < d_o(f_k, f) < \delta / 2^n,
\]
and we obtain
\[
p_n(f_k, f) < \frac{\delta}{1 - \delta} < \epsilon.
\]

Finally we have
\[
|f_k(z) - f(z)| \leq p_n(f_k, f) < \epsilon
\]
for all \(k\) greater than a certain \(k_o\) and \(z \in K\). It follows that the convergence is uniform on \(K\).

(**) \(\Rightarrow\) (*):

Given any \(\epsilon > 0\), we can choose an integer \(n\) such that
\[
\sum_{k=n+1}^{\infty} 1/2^k < \epsilon / 2.
\]
Furthermore we take a small $\delta > 0$ such that $\frac{\delta}{1-\delta} < \varepsilon/2$.

By the uniform convergence on a compact set $K_n$,

$$|f_k(z) - f(z)| < \delta \quad (k \geq k_0, z \in K_n).$$

Then

$$p_n(f_k, f) \leq \delta, \quad \frac{p_n(f_k, f)}{1 + p_n(f_k, f)} \leq \frac{\delta}{1-\delta} < \varepsilon/2.$$ 

We note $p_m(f_k, f) \leq p_n(f_k, f)$ when $1 \leq m \leq n$. Now it follows that

$$d_o(f_k, f) \leq \sum_{n=1}^{\infty} \frac{1}{2^m} \cdot \frac{p_m(f_k, f)}{1 + p_m(f_k, f)} + \sum_{m=n+1}^{\infty} \frac{1}{2^m} < \frac{\varepsilon}{2} \sum_{m=1}^{n} \frac{1}{2^m} + \frac{\varepsilon}{2} < \varepsilon$$

for all $k \geq k_0$, and the lemma is proved.

Let us define a function $F: \Omega \rightarrow \mathcal{F}(\Omega)$ as follows:

$$F(a) = f_a.$$ 

It is clear that

**LEMMA 2.** $F$ is one-to-one mapping of $\Omega$ onto $\mathcal{F}(\Omega)$.

We now prove

**LEMMA 3.** $F$ is continuous mapping on $\Omega$.

**PROOF.** Since $\Omega$, and $\mathcal{F}(\Omega)$ are metric space it is sufficient to show that the sequence $\{ F(a_n) \}$ converges to $F(a_o)$ in $\mathcal{F}(\Omega)$ for any sequence $\{ a_n \}$ of points $\Omega$ which converges to a point $a_o \in \Omega$. We let denote

$$F(a_n) = f_n, \quad f_o(a_n) = b_n, \quad (n = 0, 1, 2, \ldots).$$ 

Since $\lim_{n \to \infty} f'_o(a_n) = f'_o(a_o) > 0$, there exists a positive integer $N$ such that

$$|f'_o(a_n)| > 0 \quad \text{for all} \quad n \geq N.$$ 

Therefore, for each integer $n \geq N$, there exists a sequence $\{ k_n \}$ such that

$$|k_n| = 1, \quad k_n f'_o(a_n) > 0, \quad \text{and} \lim_{n \to \infty} k_n = 1.$$ 

For example, $k_n = \exp (-i \arg f'_o(a_n))$, $(n \geq N)$.

We can find the fact

**(A).** For each $n \geq N$, $k_n f_o$ is an analytic function of $\Omega$ onto $D$, and the sequence $\{ k_n f_o \}$ converges to $f_o$ uniformly in the wider sense on $\Omega$.
Now for each $n \geq N$, we put $c_n = k_nb_n$, and consider an analytic function $g_n$ of $D$ onto itself such that

$$g_n(c_n) = 0, \quad g_n'(c_n) > 0.$$  

Let us consider the composed function $h_n = g_n(k_nf_o)$, for each $n \geq N$. Figure (*) shows the relations of these functions. Then, each $h_n$ is an analytic function of $\Omega$ onto $D$.

Moreover,

$$h_n(a_n) = g_n(k_nf_o(a_n)) = g_n(c_n) = 0,$$

$$h_n'(a_n) = g_n'(k_nf_o(a_n)) \cdot k_nf_o'(a_n) = g_n'(c_n) \cdot k_nf_o'(a_n) > 0.$$

Fig. (*).  

By the uniqueness of the Riemann Mapping Theorem, we conclude that

$$h_n = f_n \quad (n \geq N) \quad (3).$$

On the other hand, it is clear that $g_n$ can be written as

$$g_n(z) = \frac{z-c_n}{1-c_nz} \quad (n \geq N),$$

and it follows that:

-22-
(B). The sequence \( \{ g_n \} \) converges uniformly to the identity function \( f_0(z) = z \), in the wider sense on \( \Omega \).

From the facts (A), (B) we know that the sequence \( \{ h_n \} \) \( (n \geq N) \) converges uniformly to \( h_0 = f_0 \) in the wider sense on \( \Omega \). It follows from (3) that the sequence \( \{ F(a_n) \} \) converges to \( F(a_0) \) in \( \mathcal{F}(\Omega) \), and so the lemma is proved.

We shall show next the continuity of \( F^{-1} \).

**LEMMA 4.** \( F^{-1} \) is continuous mapping on \( \mathcal{F}(\Omega) \).

**PROOF.** We recall that \( \mathcal{F}(\Omega) \) is metric space and convergence of a sequence in \( \mathcal{F}(\Omega) \) is uniform in the wider sense on \( \Omega \). Then it is sufficient to show the sequential continuity of \( F^{-1} \).

Let \( \{ f_n \} \) be a sequence of elements of \( \mathcal{F}(\Omega) \) which converges to \( f_0 \in \mathcal{F}(\Omega) \).

Let \( a_0 = F^{-1}(f_0) \).

By the application of Hurwitz's Theorem [1] we conclude that evey sufficiently small neighborhood \( U \) of \( a_0 \) contains exactly one zero of each \( f_n \) for each \( n \geq N \).

Denote \( a_n = F^{-1}(f_n) \), \( (n = 1, 2, 3, \ldots ) \).

Then each \( a_n \) is a zero of \( f_n \), and by the Riemann Mapping Theorem the only zero. Hence, if \( n \geq N \), \( a_n \in U \), i.e.,

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} F^{-1}(f_n) = a_0,
\]

and the continuity of \( F^{-1} \) is proved.

From these lemma, we obtain the following theorem:

**THEOREM.** \( \Omega \) and \( \mathcal{F}(\Omega) \) are homeomorphic.
REFERENCES

